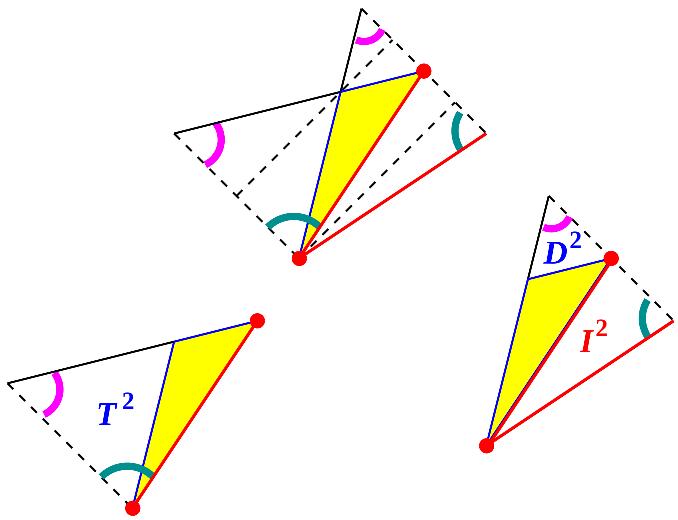
Relativity and geometry in (flat) spacetime



Walter Kohn Lecture

Sherbrooke, 6 November 2007

How to construct Minkowski Diagrams (1908)

directly from Einstein's postulates (1905)

as an exercise in *Plane geometry* in (flat) spacetime.

Euclid vs. Descartes

Light rectangles

Einstein's Two Postulates (Voraussetzungen) (1905)

1. In electrodynamics, as well as in mechanics, no properties of phenomena correspond to the concept of absolute rest.

...dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen....

2. Light always propagates in empty space with a definite velocity c, independent of the state of motion of the emitting body.

...sich das Licht im leeren Raume stets mit einer bestimmten, von Bewegungszustande des emittierenden Körpers unabhängigen Geschwindigkeit V fortpflanze.

Einstein's Third Postulate (1905)

3. If a clock at A runs synchronously with clocks at both B and C, then the clocks at B and C also run synchronously relative to each other.

Wenn die Uhr in A sowohl mit der Uhr in B als auch mit der Uhr in C synchron läuft, so laufen auch die Uhren in B und C synchron relativ zueinander.

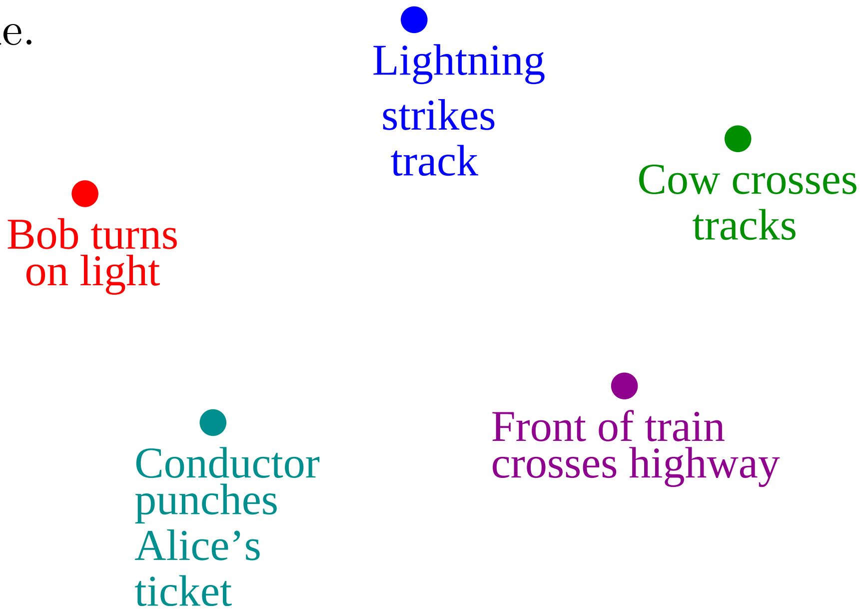
3′. If event A is simultaneous with event B and event C, then events B and C are also simultaneous.

3″. If an event A happens in the same place as event B and event C, then the events B and C also happen in the same place.

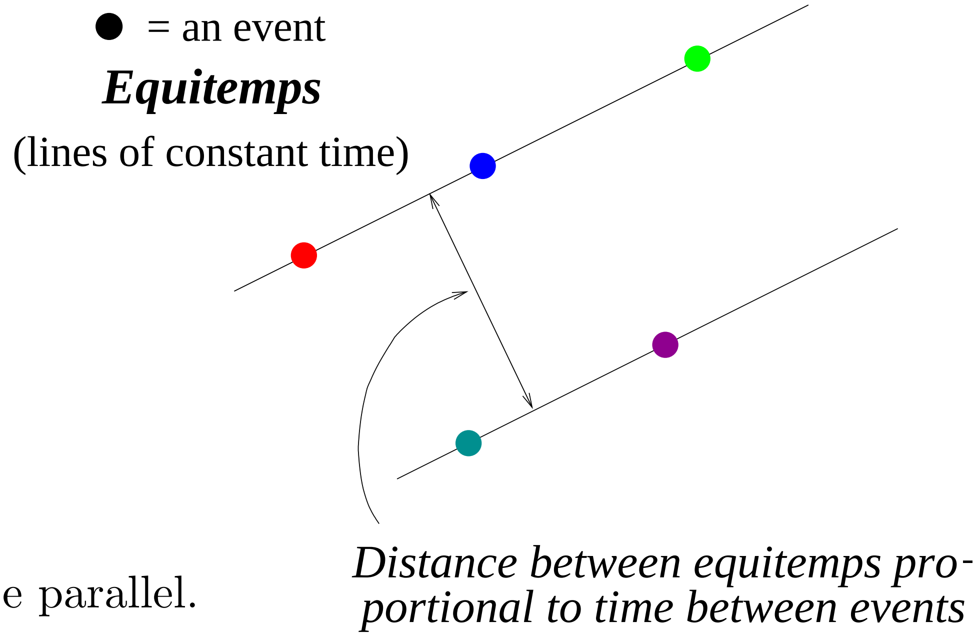
An event:

Something happening at definite place and time. Represented by a point in spacetime.

Alice makes a plane diagram depicting events at various times and places in one spatial dimension (e.g. along a long straight railroad track).



Alice organizes events in her diagram by time:
Simultaneous events placed on single straight line



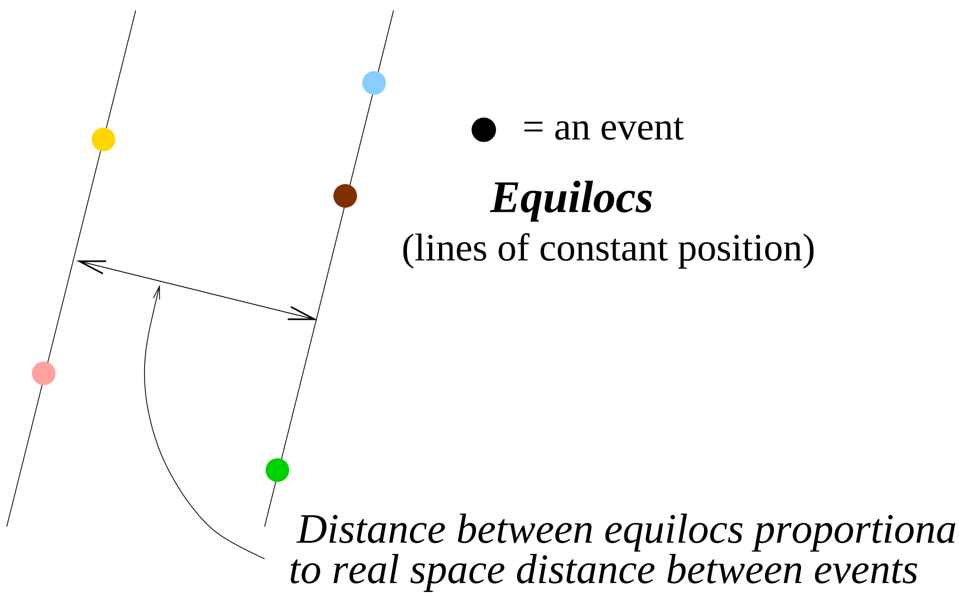
Equitemps must be parallel.

Alice slides events along equitemps
to further organize them by location:

Events in same place lie on same straight line

Equilocs must be parallel.

Can't be parallel to equitemps, but otherwise orientation arbitrary.



Alice redefines the foot:

1 conventional foot (ft) = 0.3048 m.

1 foot (f) = 0.299792458 m.

1 f/ns = 299,792,458 m/s = c, speed of light.
(ns = nanosecond = 10^{-9} sec)

Alice relates spatial and temporal scales:

Equilocs representing events 1 f apart are same distance λ apart in diagram as equitemps representing events 1 ns apart.

Some of Alice's equitemps and equilocs and her scale factor λ

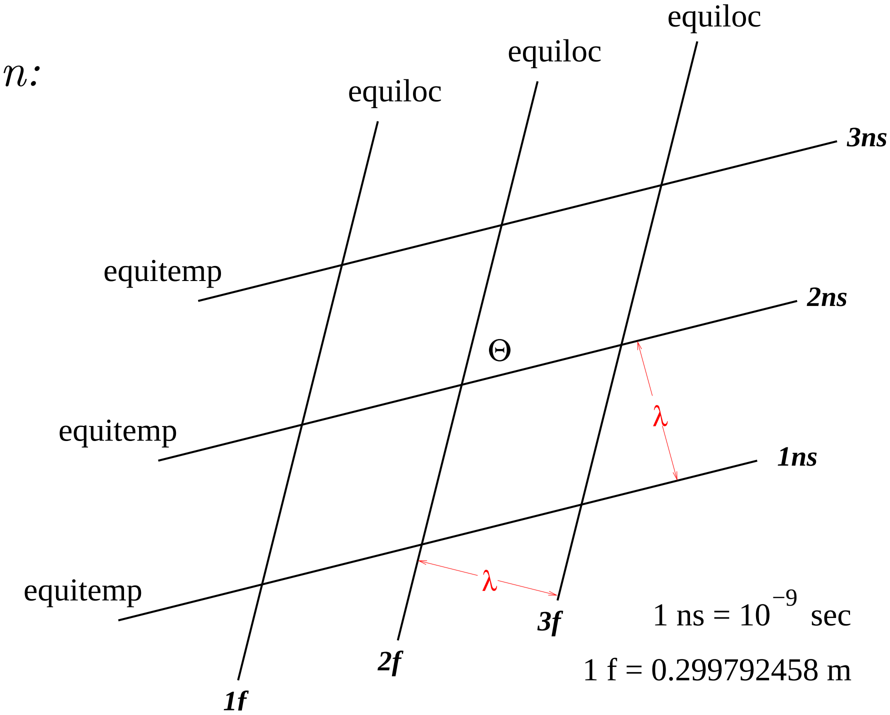
Conventional orientation:

Equilocs more vertical than horizontal.

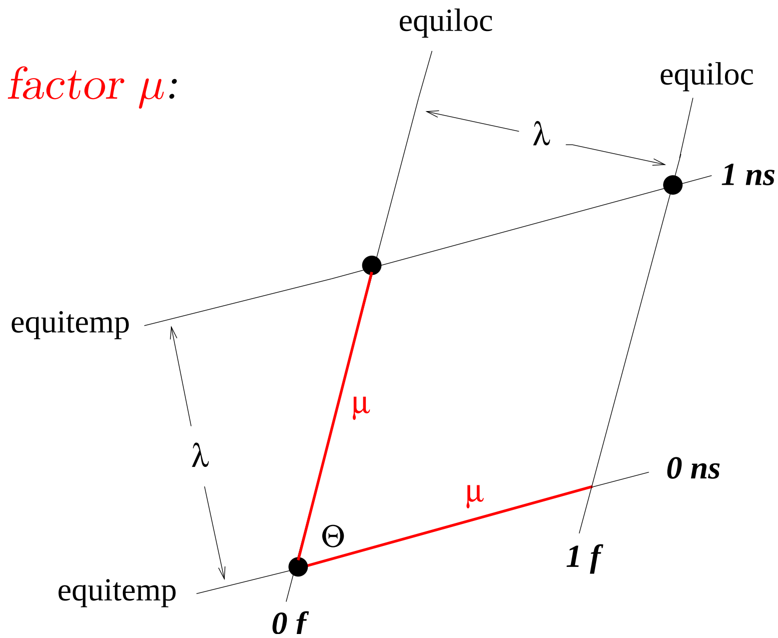
Equitemps more horizontal than vertical.

Both symmetrically disposed about 45° lines.

Time increases with height on page.



Alternative scale factor μ:



Equilocs and equitemps are characterized
by two independent parameters: any two of λ, μ, Θ

Area of *unit rhombus* $= \lambda\mu = \mu^2 \sin\Theta$.

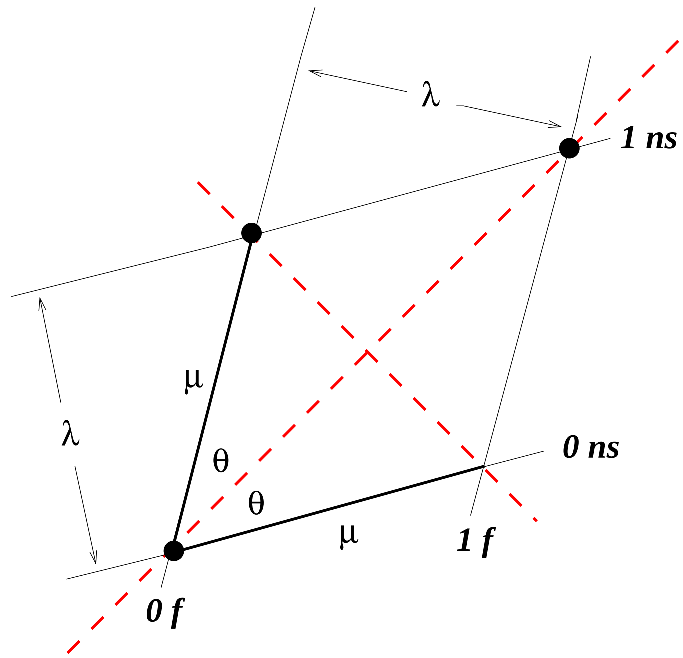
Photon trajectory:

All events in the history of something moving at 1f/ns

Photon trajectories *bisect* angle $\Theta = 2\theta$ between equilocs and equitemps

(Equilocs and equitemps *symmetrically disposed* about photon trajectories)

Trajectories of oppositely moving photons are *perpendicular*.



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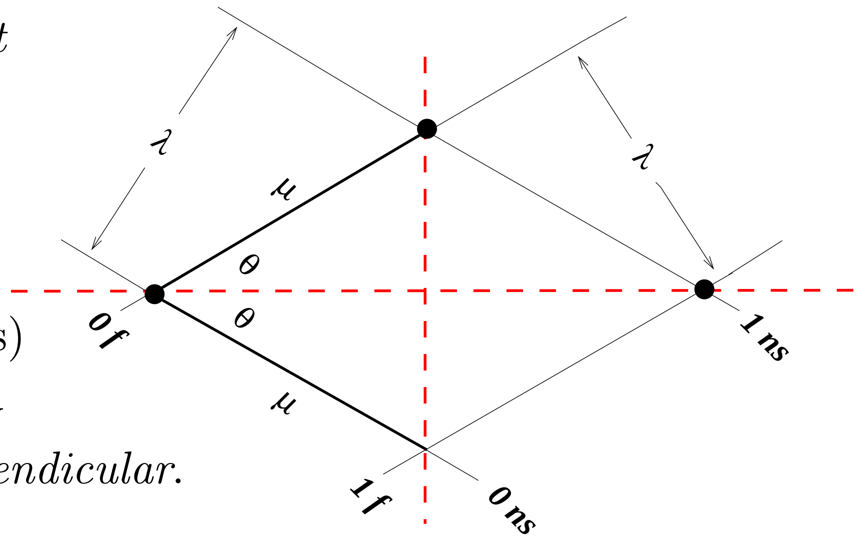
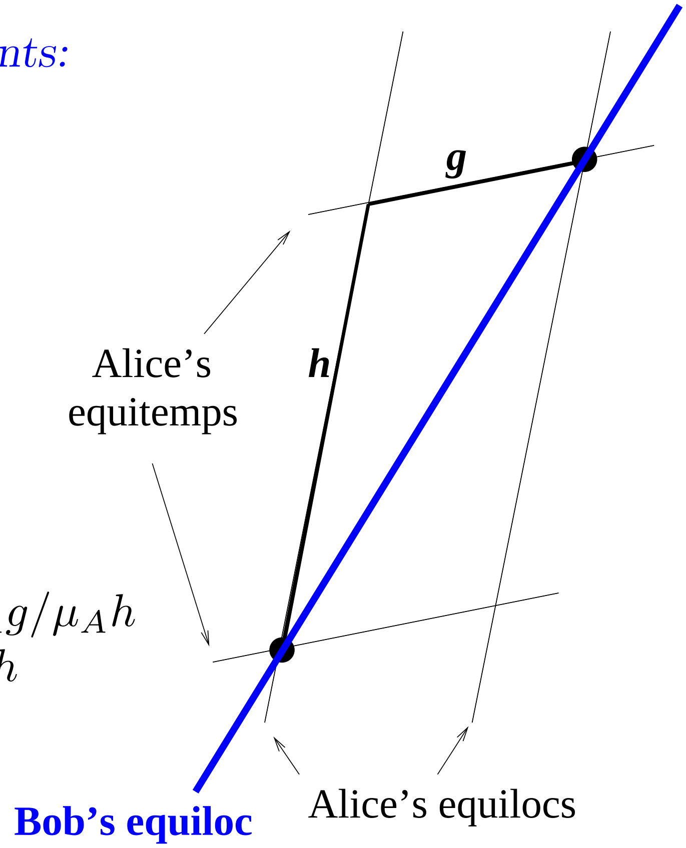


Bob's description of the same events:

Bob moves uniformly with respect to Alice.

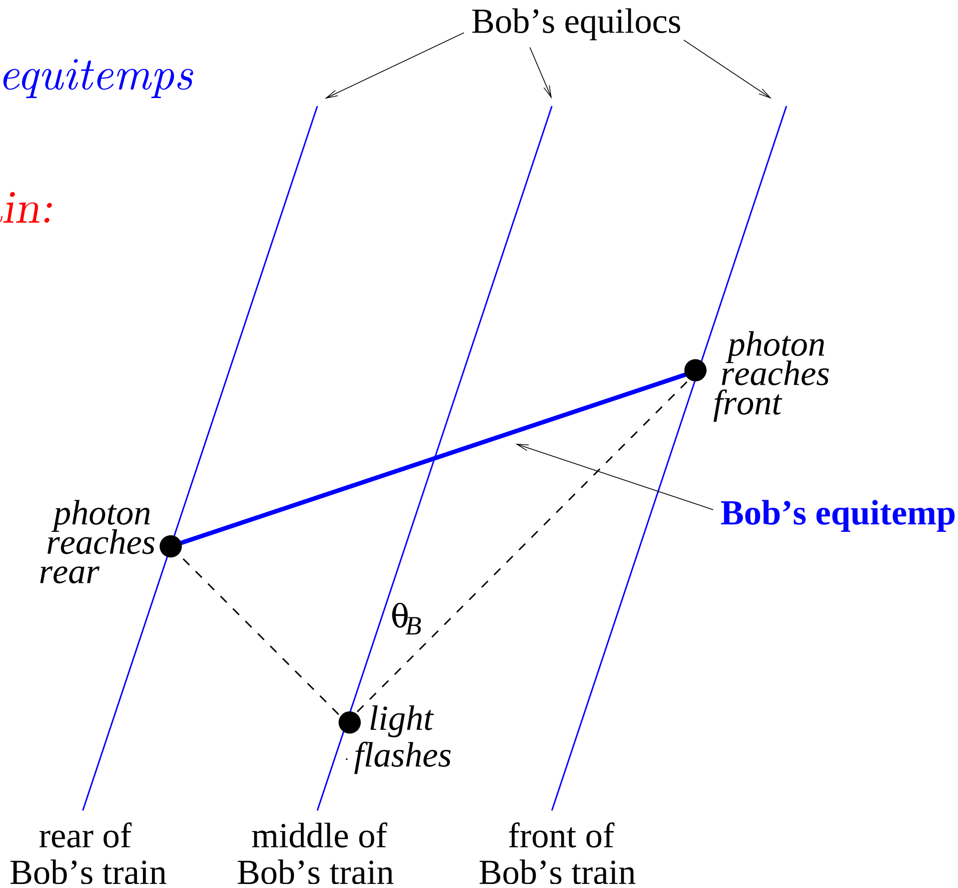
He uses Alice's diagram to depict events, but tries to impose on it *his own* equilocs and equitemps.



$$v_{BA} = \mu_A g/\mu_A h$$
$$= g/h$$

Determining Bob's *equitemps* in Alice's diagram.

Einstein's Train:



Determining Bob's *equitemps* in Alice's diagram.

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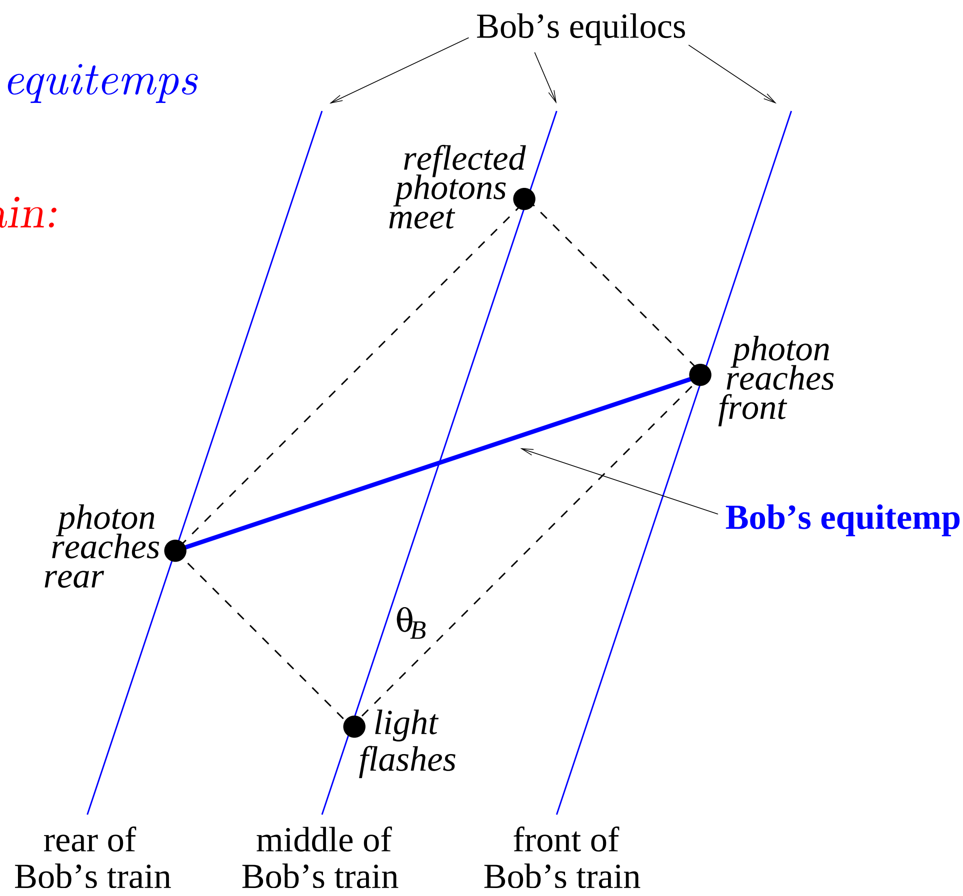

Determining Bob's *equitemps*
in Alice's diagram.

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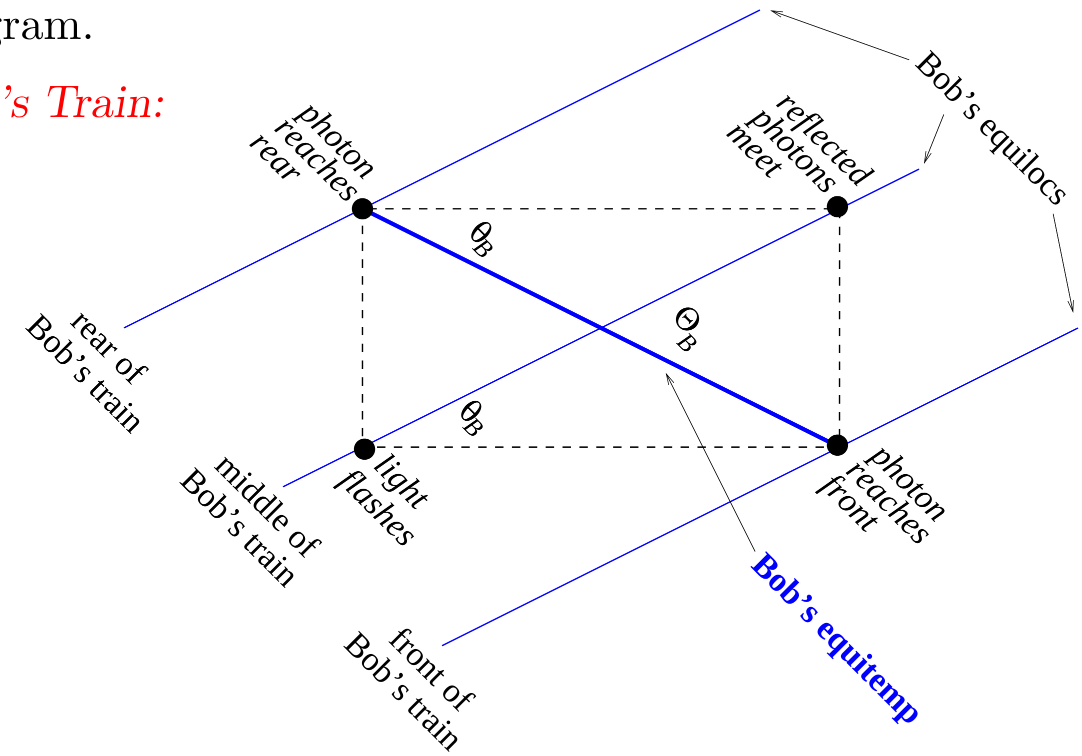
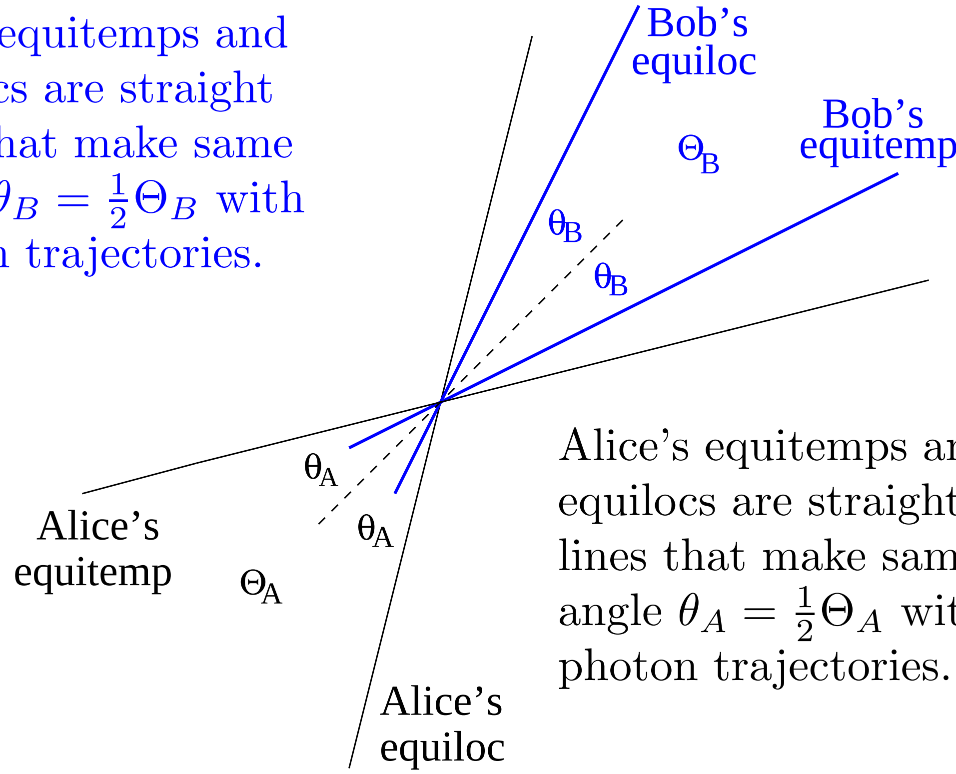


Bob's equitemps and equilocs are straight lines that make same angle $\theta_B = \frac{1}{2}\Theta_B$ with photon trajectories.



Alice's equitemps and equilocs are straight lines that make same angle $\theta_A = \frac{1}{2}\Theta_A$ with photon trajectories.

Cannot tell who made the diagram first and who later added their own equitemps and equilocs.

Einstein (1905):

The second principle is
only apparently incompatible
with the first.

nur scheinbar unverträgliche

It remains only to determine the relation between Alice's scale factors λ_A, μ_A and Bob's, λ_B, μ_B

Independent of relation between scale factors:

Relativity of simultaneity (quantitative)

$v_{BA} = g/h$

Bob: **P**, **R** at same place

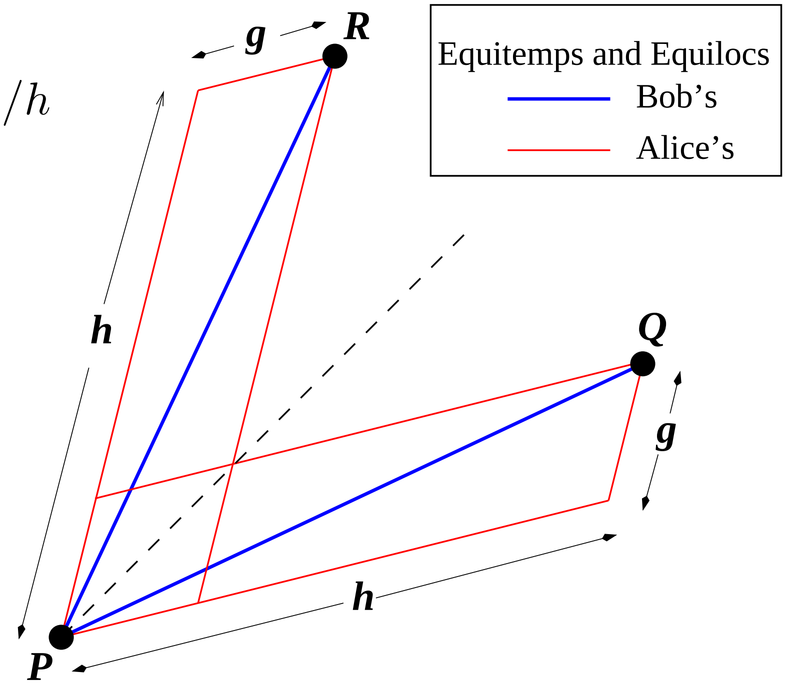
P, **Q** at same time

Alice: $D_{PR} = v_{BA} T_{PR}$

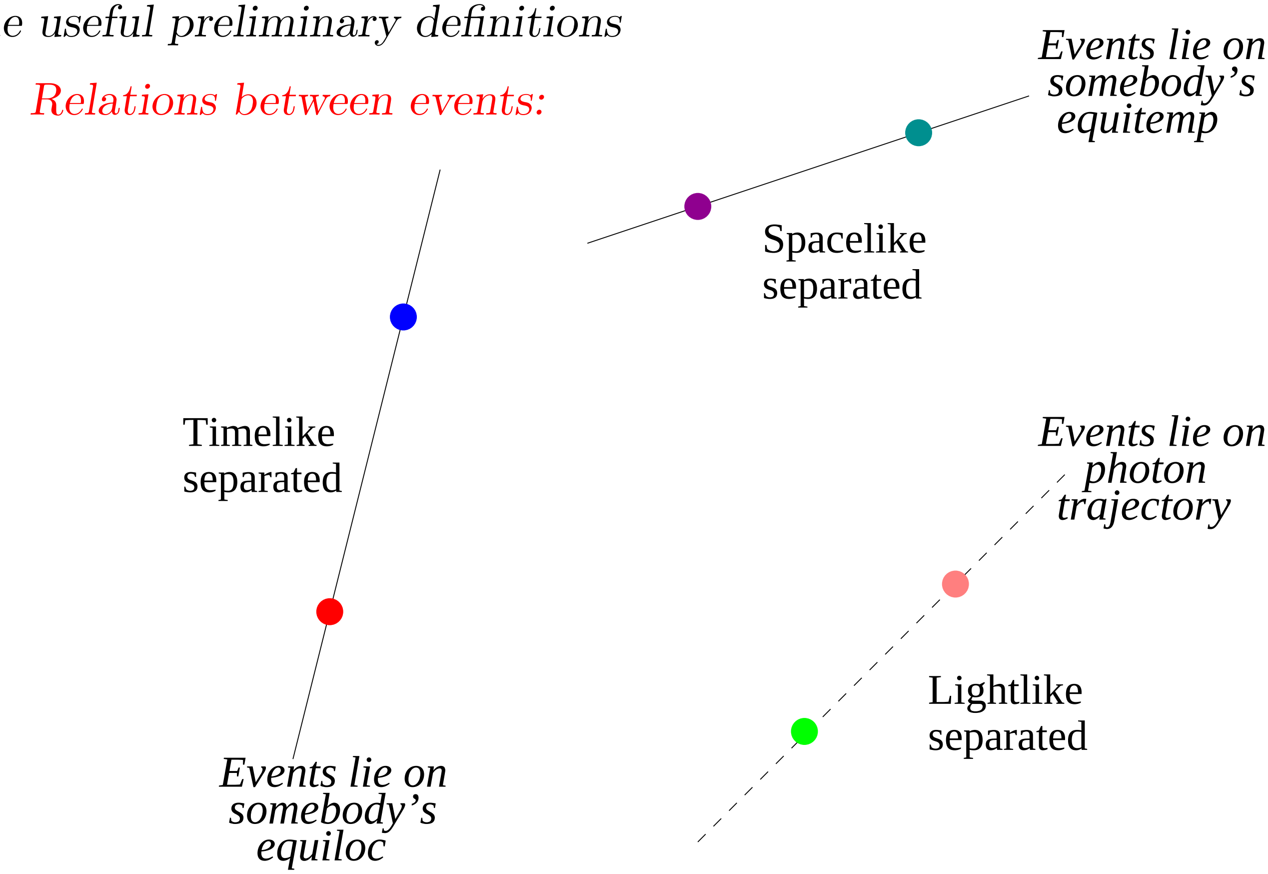
$(\mu_A g) \qquad (\mu_A h)$

$T_{PQ} = v_{BA} D_{PQ}$

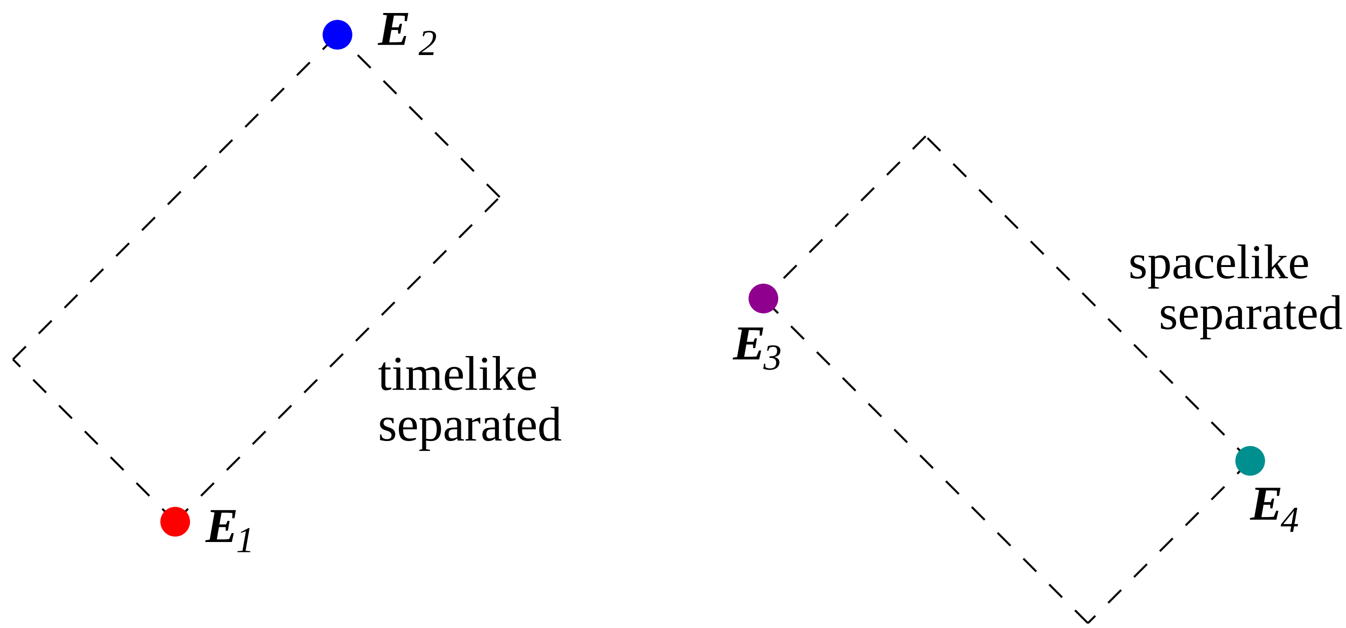
$(\mu_A g) \qquad (\mu_A h)$



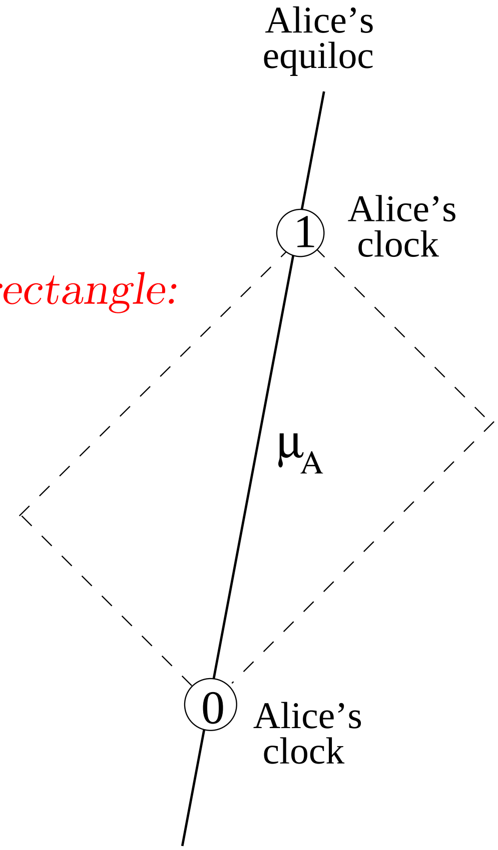
Some useful preliminary definitions



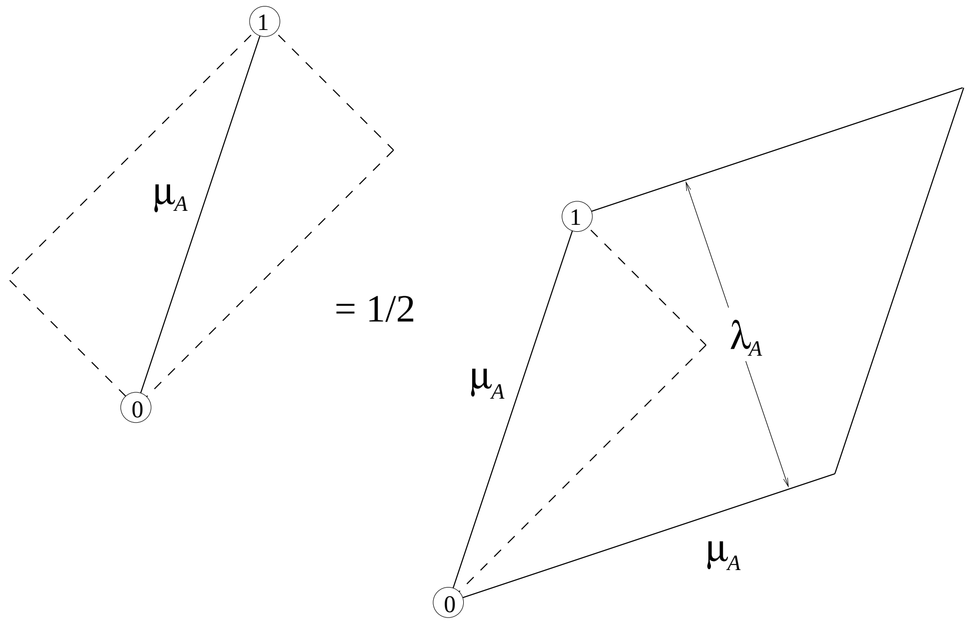
Two events determine a *light rectangle*.



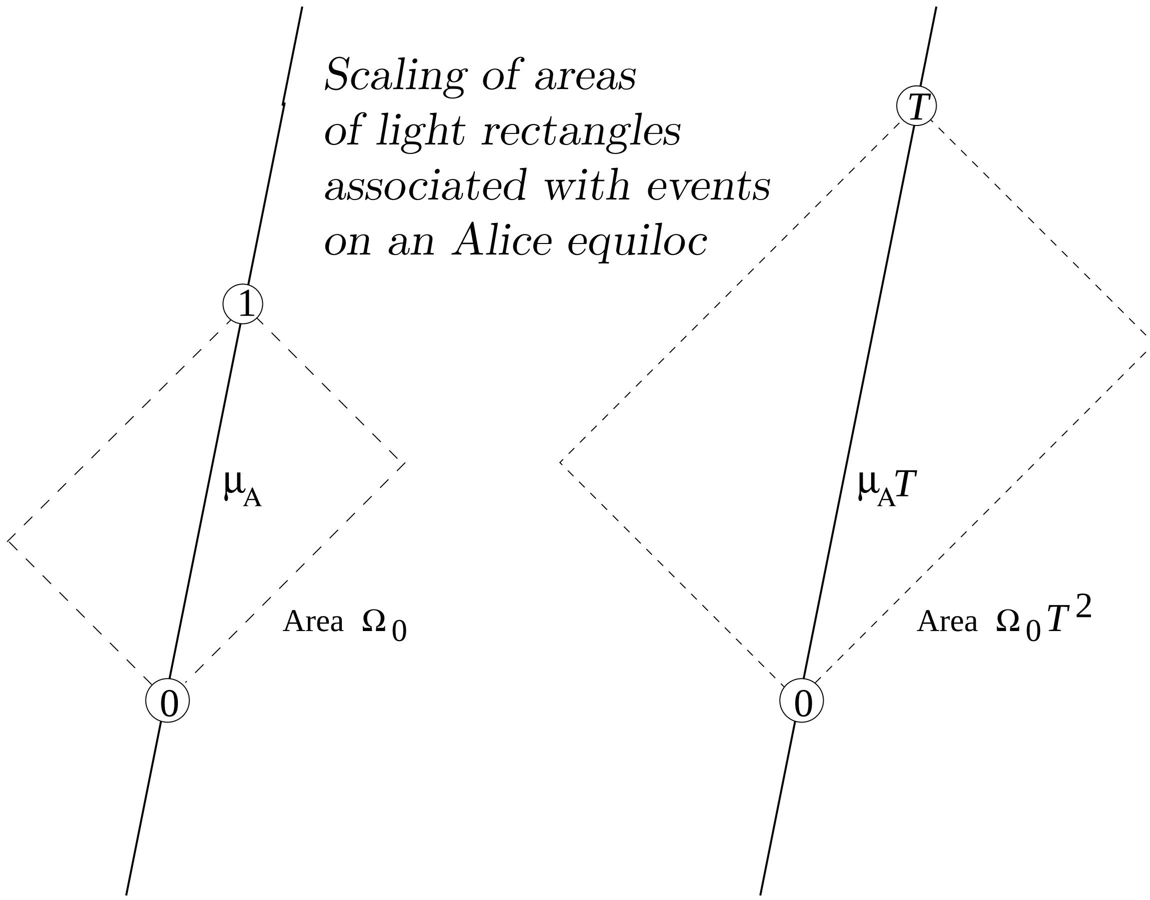
Alice's unit light rectangle:



Area Ω_0 of Alice's unit light rectangle



$$\Omega_0 = \frac{1}{2}\lambda_A \mu_A$$


Scaling of areas
of light rectangles
associated with events
on an Alice equiloc
1
0
μA
Area Ω0
T
0
μAT
Area Ω0T2

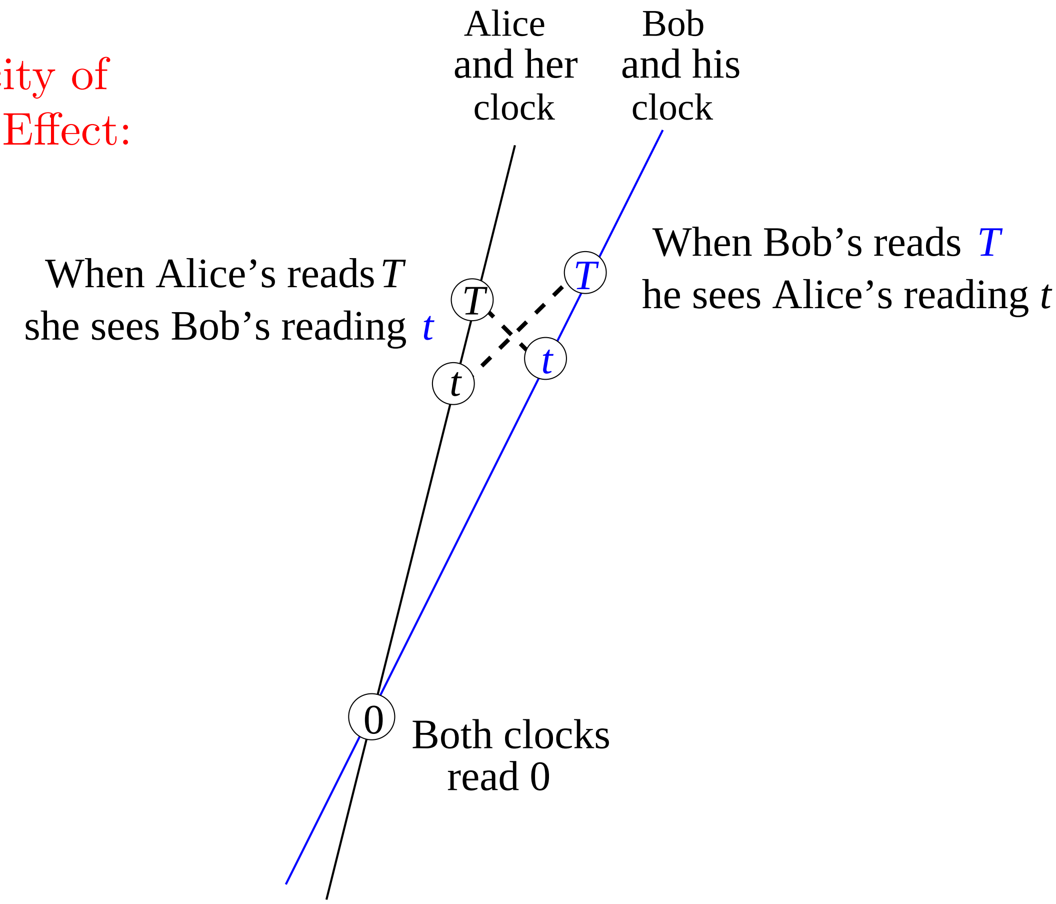
Relation between Alice's and Bob's scale factors determined by reciprocity of the Doppler effect:

When Alice, Bob, and their clocks are all together they both set their clocks to 0.

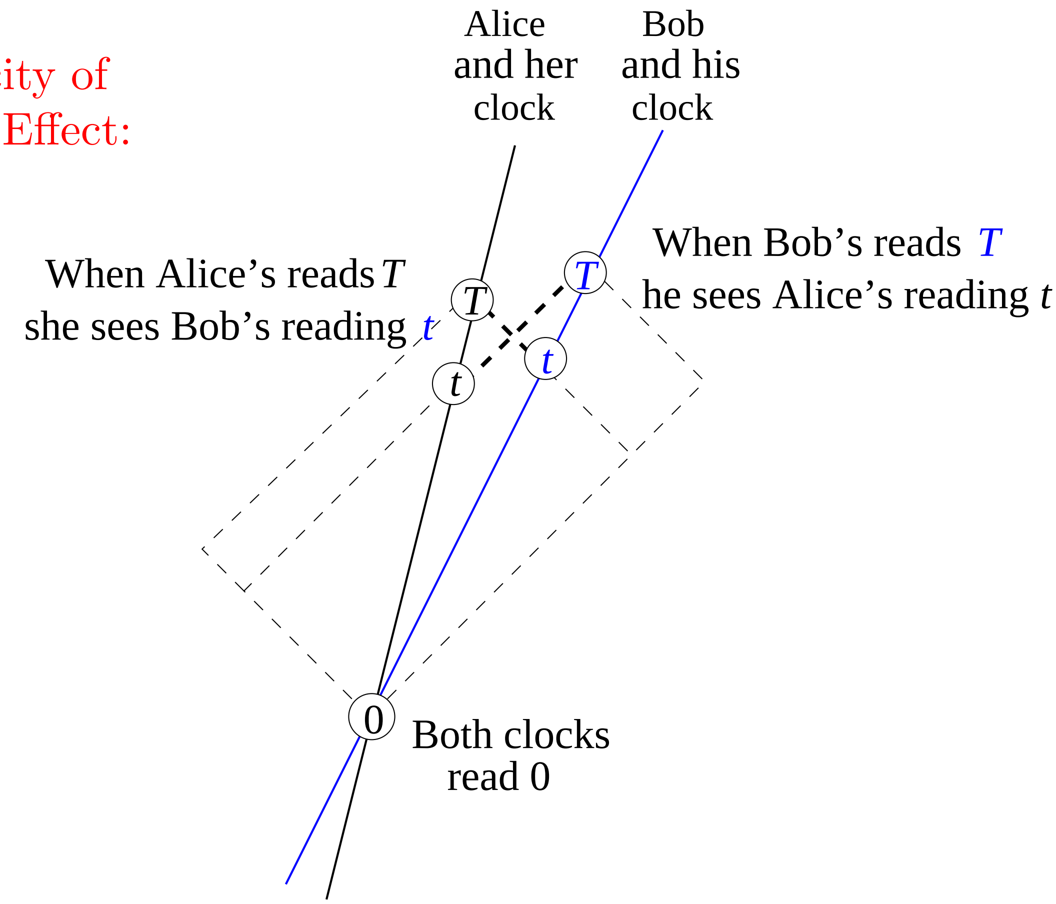
Later, when Alice's clock reads T she looks at Bob's. She sees Bob's clock reading t.

When Bob's clock reads same T he looks at Alice's. He must see Alice's clock reading same t.

Reciprocity of Doppler Effect:



Reciprocity of Doppler Effect:

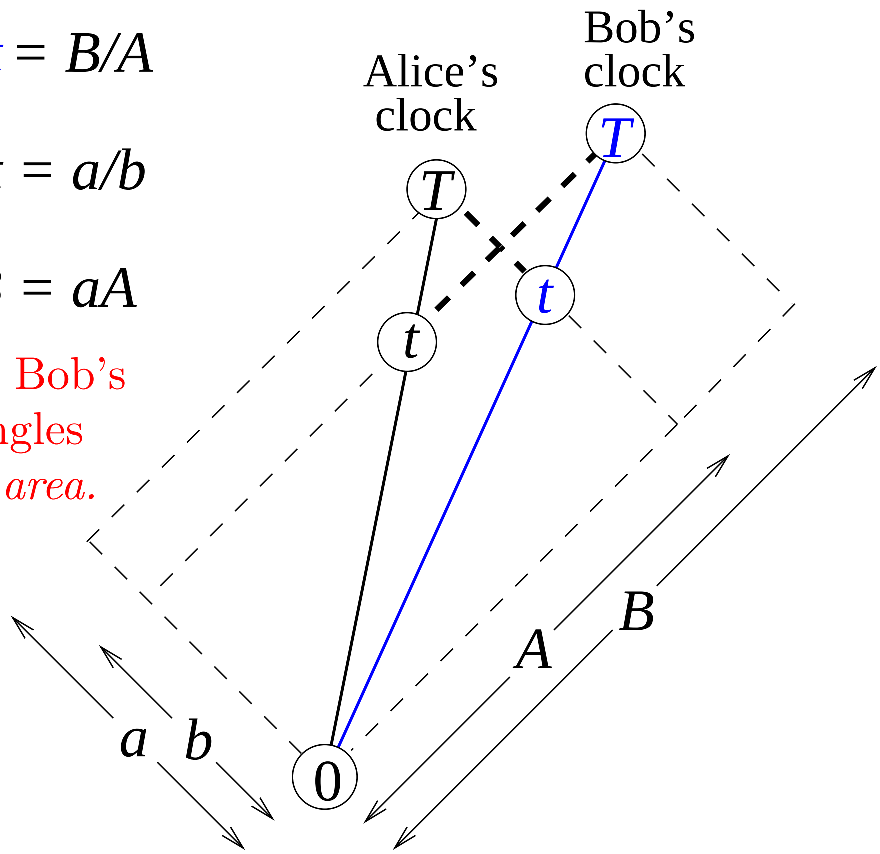


$T/t = B/A$

$T/t = a/b$

$bB = aA$

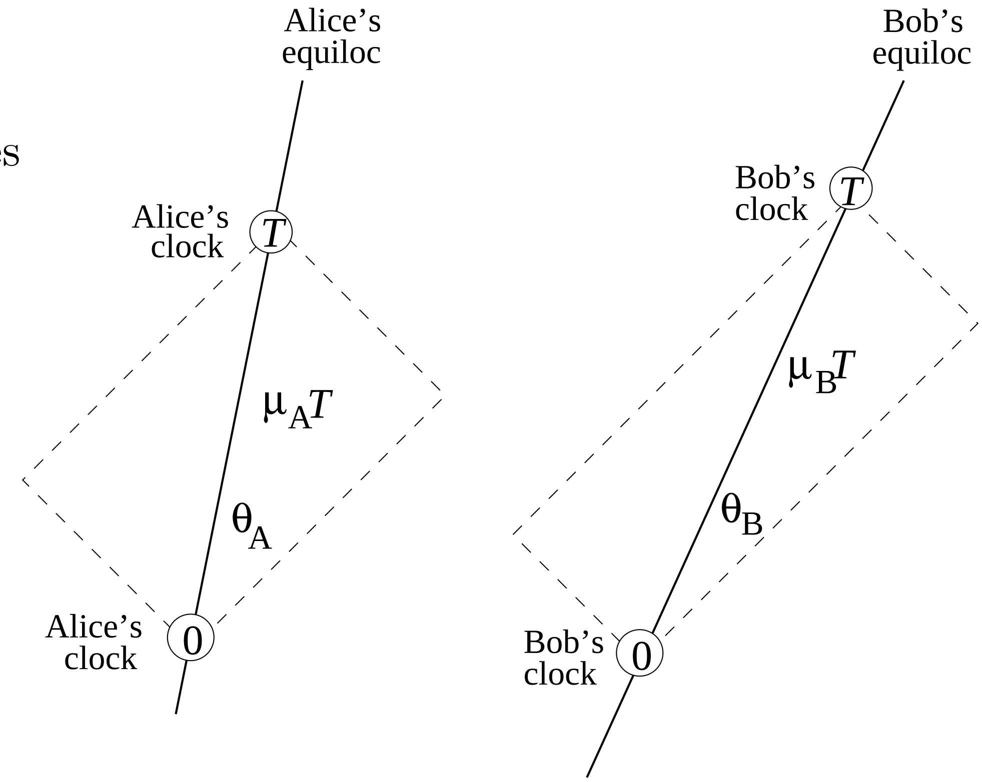
Alice's and Bob's light rectangles have *same area.*



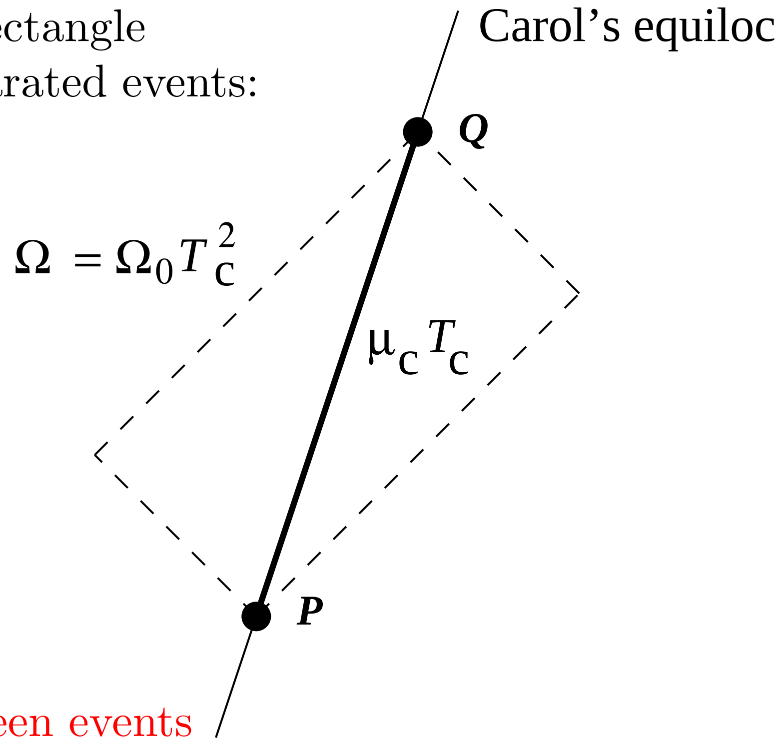
$T = 1 \Longrightarrow$
unit light rectangles
have same area.
$\Omega_0 = \frac{1}{2}\mu\lambda$

Product $\mu\lambda$
of scale factors
is the same for
everyone:

$\mu_A\lambda_A = \mu_B\lambda_B$



Meaning of area Ω of light rectangle
for *any* pair of time-like separated events:

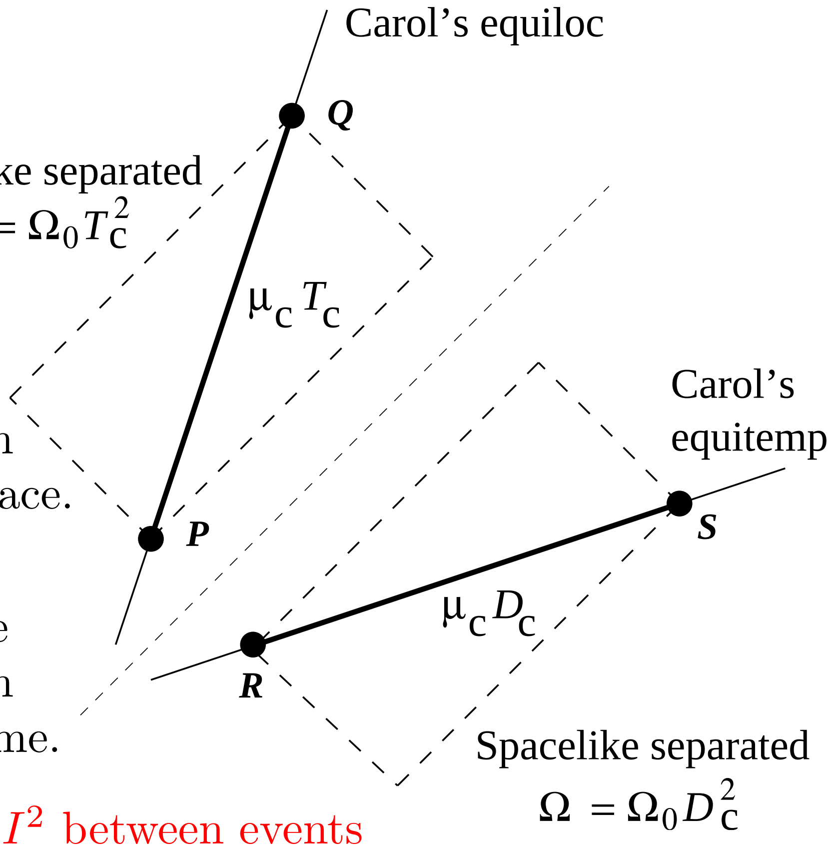


Ω/Ω_0 is square of time between events
in frame in which they are at same place.

Meaning of area Ω
of light rectangle
for *any* pair of events:

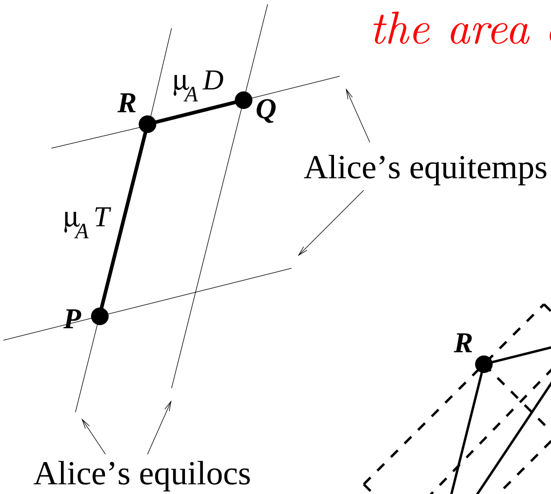
Timelike separated:
Ω/Ω_0 is square of time between events in frame in which they are at same place.

Spacelike separated:
Ω/Ω_0 is square of distance between events in frame in which they are at same time.



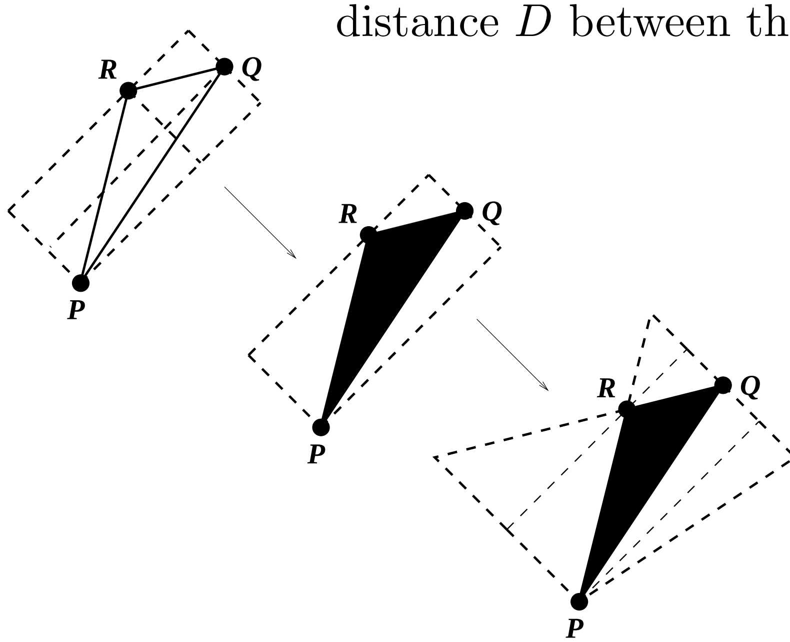
Ω/Ω_0 *is squared interval* I^2 between events

The squared interval between two events is proportional to the area of the light rectangle they determine.



What about $I^2 = |T^2 - D^2|$?

Interval I between events P and Q in terms of Alice's time T and distance D between them.



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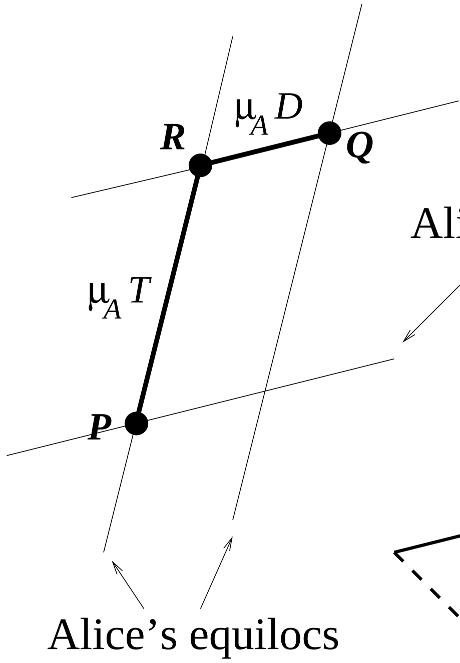
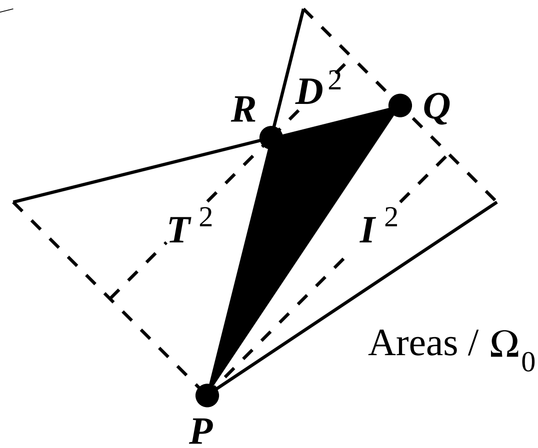
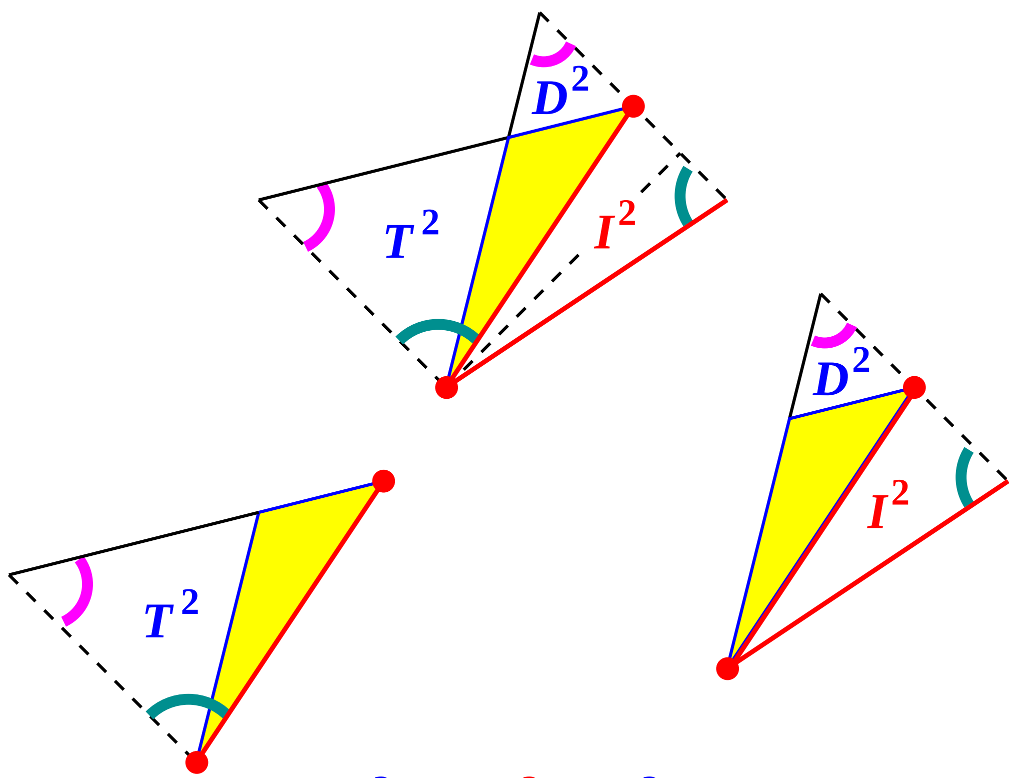






$$T^2 = I^2 + D^2$$

Application (in 3+1 dimensions)

*How to measure the interval between P and Q using only light signals and a single clock:**

Alice moves uniformly with her clock;

Alice and her clock are both present at P.

Bob is present at Q.

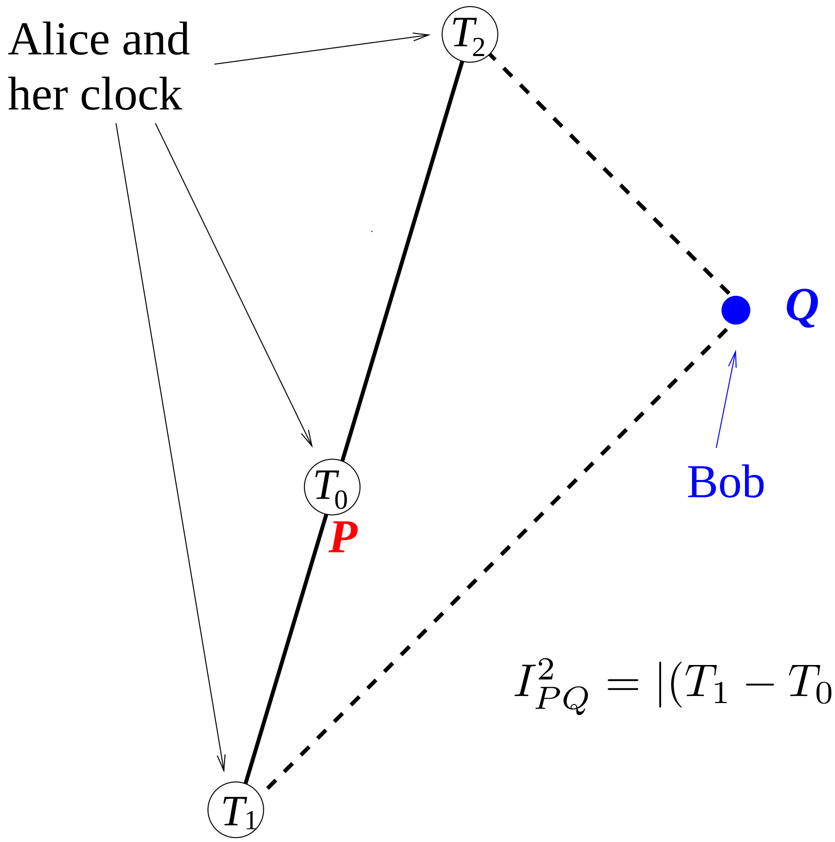
When P happens Alice's clock reads T_0.

When Q happens, Bob *sees* Alice's clock reading T_1.

When Alice *sees* Q happen, her clock reads T_2.

$$I_{PQ}^2 = |(T_1 - T_0)(T_2 - T_0)|$$

*Robert F. Marzke, 1959 Princeton senior thesis.



$$I_{PQ}^2 = |(T_1 - T_0)(T_2 - T_0)|$$

P and Q
spacelike separated

$\Omega_{T_2,T_0} = ab$

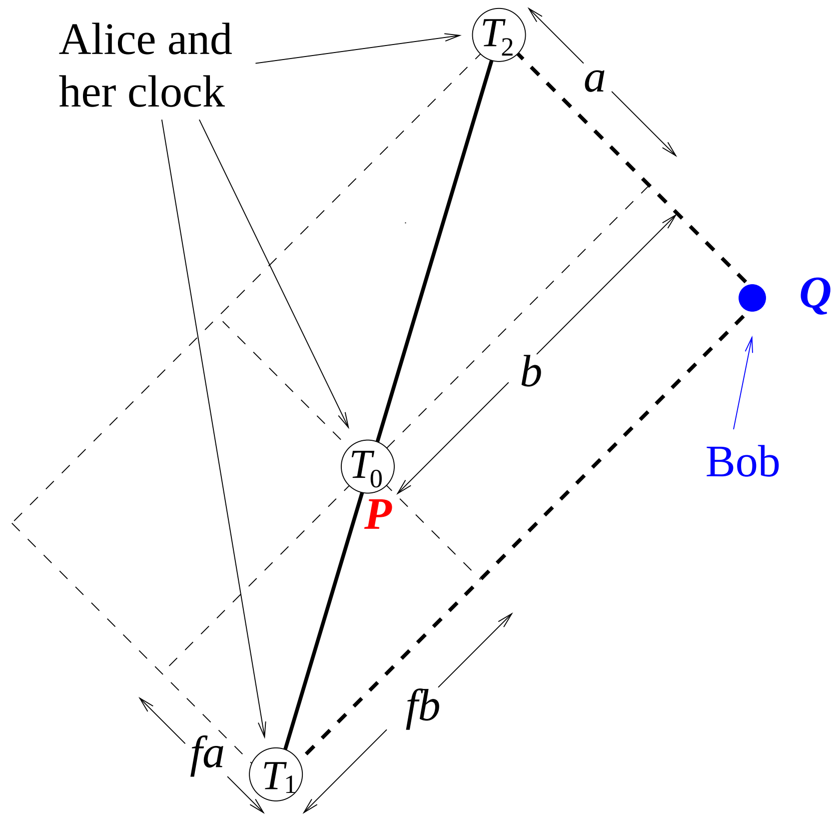
$\Omega_{T_1,T_0} = f^2 ab$

$\Omega_{P,Q} = fab$

$$\Longrightarrow$$

$$\Omega^2_{P,Q} \;=\; \Omega_{T_2,T_0}\;\Omega_{T_1,T_0}$$

$$I^2_{P,Q} = (T_2 - T_0)(T_0 - T_1)$$



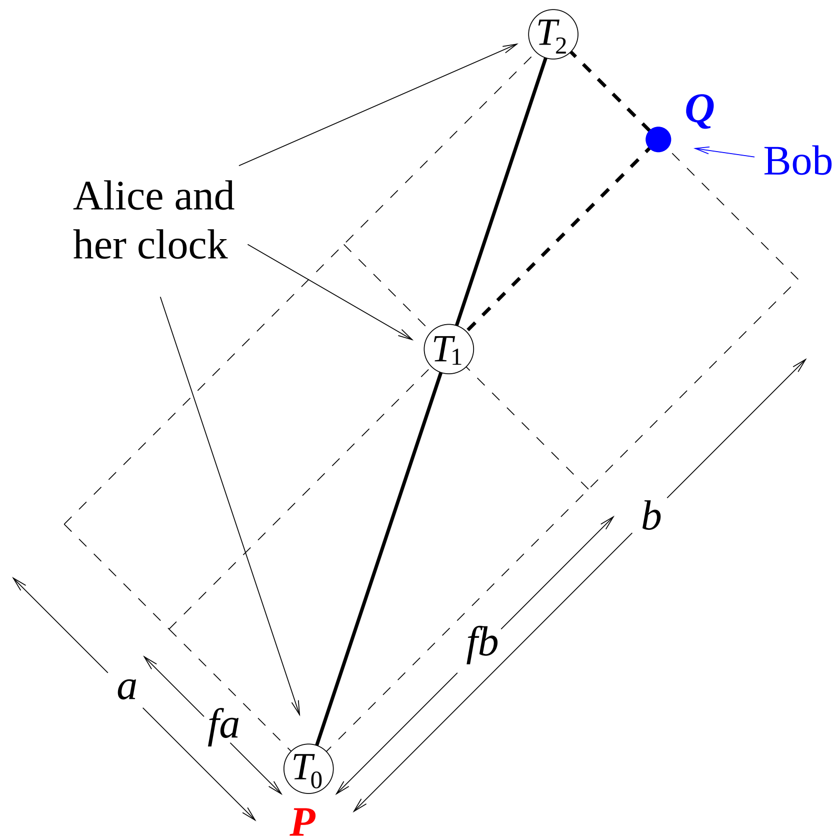
P and *Q*
timelike separated

$\Omega_{T_2,T_0} = ab$
$\Omega_{T_1,T_0} = f^2ab$
$\Omega_{P,Q} = fab$

$$\Longrightarrow$$

$$\Omega^2_{P,Q} \;=\; \Omega_{T_2,T_0}\;\Omega_{T_1,T_0}$$

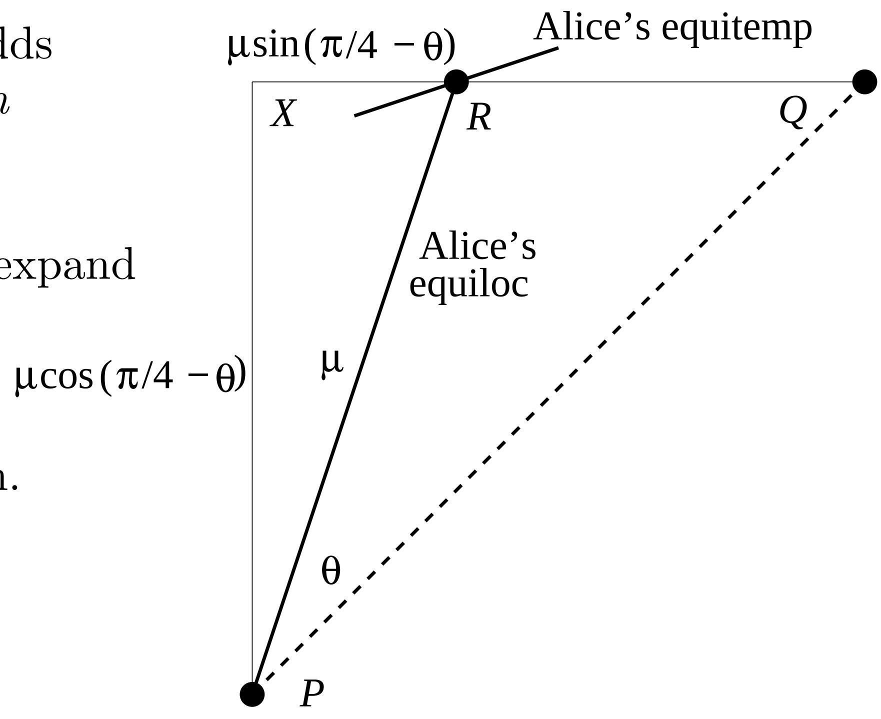
$I^2_{P,Q} = (T_2 - T_0)(T_0 - T_1)$



Stacking plane diagrams
in orthogonal direction.

Isotropy: When Alice adds *second spatial dimension* perpendicular to plane, photon trajectories through a point should expand to right circular cone.

Sets scale factor σ for perpendicular dimension.

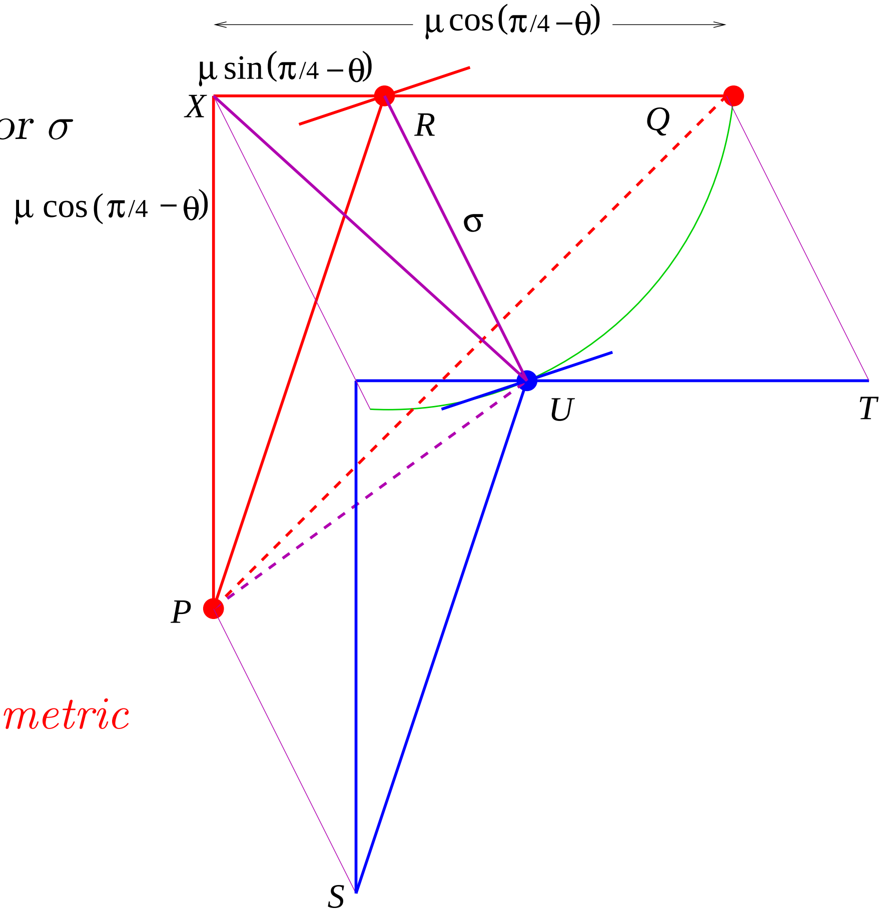


Determination of perpendicular scale factor σ

$$\sigma^2 + \mu^2 \sin^2(\pi/4 - \theta) = \mu^2 \cos^2(\pi/4 - \theta)$$

$$\begin{aligned}\sigma^2 &= \mu^2 \cos(\pi/2 - 2\theta) \\ &= \mu^2 \sin(2\theta) \\ &= \mu^2 \sin \Theta \\ &= \mu\lambda\end{aligned}$$

σ is (invariant) geometric mean of μ and λ.



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