$$|n_k\rangle = \frac{(a_k^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$H = \sum_k \left(\frac{1}{2} + a_k^\dagger a_k\right)\hbar\omega_k$$

$$[a_k, a_k^\dagger] = 1$$

$$\sum_i \left(\frac{P_i^2}{2m} + \frac{1}{2}m\omega_i^2 X_i^2\right)$$

$$|x\ y\rangle = \psi^{+}(x)\,\psi^{+}(y)\,|0\rangle$$

$$[\psi(x), \psi^{+}(y)] = \delta(x-y)$$

$$H = \int dx\ \underline{\psi^{+}(x)}\left(-\frac{\hbar^2}{2m}\nabla^2\right)\underline{\psi(x)}$$

$$X, P \longrightarrow \psi^{+}(x) \longrightarrow \text{opérateur de création de ptcle.}$$

$$(\vec{E}(x), \vec{B}(x))$$

Sénéchal. Ch. 4

Ch. 5.3 Espace de Fock

1. Etats sym. pour bosons.
2. Opérateurs création annihilation.
3. Etats antisym. (fermions)
4. ψ^+, ψ
5. Opérateur densité
6. Changement de base

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États symétrisés :

$$|\psi\rangle = \int dx\, \boxed{\psi(x)}\, |x\rangle$$

$$= \int dx\, |x\rangle \boxed{\langle x | \psi \rangle}$$

$$\boxed{\langle x | x' \rangle = \delta(x - x')}$$

Espace de Hilbert V_1

Pour n particules

$$V_n = V_1 \otimes V_1 \otimes V_1 \otimes \dots$$

Bosons identiques

$$|x, y\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle |y\rangle + |y\rangle |x\rangle \right)$$

$V_1 \quad V_1$

Normalisation

$$\langle x', y' | x, y \rangle = \delta(x - x')\, \delta(y - y') + \delta(x - y')\, \delta(x' - y)$$

$$\frac{1}{2} \left(\langle x'| \langle y'| + \langle y'| \langle x'| \right) \left(|x\rangle |y\rangle + |y\rangle |x\rangle \right)$$

$$= \left(\delta(x - x')\, \delta(y - y') + \delta(x' - y)\, \delta(y' - x) \right)$$

Fermeture:

$$\frac{1}{2}\int dx\,dy\ |x\,y\rangle\langle x,y| = 1$$

$$\varphi(x,y) = \langle x,y|\varphi\rangle$$

$$|\varphi\rangle = \frac{1}{2}\int dx\,dy\ \varphi(x,y)\,|x,y\rangle$$

Généralisation:

$$\boxed{|x_1\ x_2 \dots x_n\rangle = \frac{1}{\sqrt{n!}}\sum_{p\in S_n} |x_{p(1)}\rangle|x_{p(2)}\rangle\dots|x_{p(n)}\rangle}$$

S_n = groupe des perm. de n objets

$$|x_1, x_2 \dots x_n\rangle = |x_{p(1)}\ x_{p(2)} \dots x_{p(n)}\rangle$$

Normalisation

$$\langle x_1\, x_2 \dots x_n | y_1\, y_2 \dots y_n\rangle = \sum_{p\in S_n} \delta(x_{p(1)} - y_1)\,\delta(x_{p(2)} - y_2)\dots\delta(x_{p(n)} - y_n)$$

$$\frac{1}{n!} \sum_{p,q \in S_n} \langle x_{p(1)} | y_{q(1)} \rangle \langle x_{p(2)} | y_{q(2)} \rangle$$

$$\cdots\cdots \langle x_{p(n)} | y_{q(n)} \rangle$$

$q(m) = i$

$$\boxed{m = q^{-1}(i)}$$

$$= \frac{1}{n!} \sum_{p,q \in S_n} \langle x_{pq^{-1}(1)} | y_1 \rangle \langle x_{pq^{-1}(2)} | y_2 \rangle$$

$$\cdots\cdots \langle x_{pq^{-1}(n)} | y_n \rangle$$

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Fermeture:

$$\Gamma = \frac{1}{n!}\int\left(\prod_{i=1}^{n} dx_i\right)|x_1 \dots x_n\rangle\langle \underbrace{x_1 \dots x_n}_{\varphi}|$$

$$\varphi(x_1 \dots x_n) = \langle x_1 \dots x_n | \varphi\rangle$$

$$|\varphi\rangle = \frac{1}{n!}\int dx_1 \dots dx_n\, \varphi(x_1 \dots x_n)|x_1 \dots x_n\rangle$$

3.2 Opérateurs de création-annihilation

Espace de Fock

$$V = \bar{V}_0 \oplus \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3 \dots$$

$\bar{V}_0 = V$ symétrisé.

Passer de $\bar{V}_n$ à $\bar{V}_{n+1}$

$$\psi^+(x)\,|x_1 \dots x_n\rangle = |x\, x_1 \dots x_n\rangle$$

↳ ajoute une particule en x et la symétrise avec les autres

$$\boxed{\psi(x)\,|x_1 \dots x_n\rangle = \sum_{i=1}^{n} \delta(x-x_i)\,|x_1 \dots \hat{x}_i \dots x_n\rangle}$$

Preuve

$$\langle y_1 \dots y_{n-1} | \psi(x) | x_1 \dots x_n \rangle$$

$$\langle \psi^{+}(x)\, y_1 \dots y_{n-1} | x_1 \dots x_n \rangle$$

$$= \langle x\, y_1 \dots y_{n-1} | x_1 \dots x_n \rangle$$

$$= \sum_{p \in S_n} \delta(y_1 - x_{p(1)})\, \delta(y_2 - x_{p(2)}) \dots \dots \delta(y_{n-1} - x_{p(n-1)})\, \delta(x - x_{\underbrace{p(n)}_{i}})$$

$$= \sum_{i=1}^{n} \delta(x - x_i) \left(\sum_{\substack{q \in S_{n-1} \\ \neq i}} \delta(y_1 - x_{q(1)})\, \delta(y_2 - x_{q(2)}) \dots \delta(y_{n-1} - x_{q(n-1)}) \right)$$

$$= \sum_{i=1}^{n} \delta(x - x_i) \langle y_1 \dots y_{n-1} | x_1 \dots \hat{x}_i \dots x_n \rangle$$

$$\psi^{\dagger}(x)\,\psi^{\dagger}(y) = \psi^{\dagger}(y)\,\psi^{\dagger}(x)$$

$$[\psi^{\dagger}(x), \psi^{\dagger}(y)] = 0$$

$$[\psi(x), \psi(y)] = 0$$

$$\boxed{[\psi(x), \psi^{\dagger}(y)] = \delta(x-y)}$$

$$+\psi(x)\,\psi^{\dagger}(y)\,|x_1 \ldots x_n\rangle = \psi(x)\,|y\,x_1 \ldots x_n\rangle$$

$$= \cancel{\sum_{i=1}^{n} \delta(x-x_i)\,|y\,x_1 \ldots \hat{x}_i \ldots x_n\rangle}$$

$$+\,\delta(x-y)\,|\hat{y}, x_1 \ldots x_n\rangle$$

$$-\,\psi^{\dagger}(y)\,\psi(x)\,|x_1 \ldots x_n\rangle$$

$$= \psi^{\dagger}(y) \sum_{i=1}^{n} \delta(x-x_i)\,|x_1 \ldots \hat{x}_i \ldots x_n\rangle$$

$$= \cancel{\sum_{i=1}^{n} \delta(x-x_i)\,|y\,x_1 \ldots \hat{x}_i \ldots x_n\rangle}$$

Exemple:

$$|0\rangle \qquad \psi^{+}(x)|0\rangle = |x\rangle$$

$$|x\, y\rangle = \psi^{+}(y)\psi^{+}(x)|0\rangle$$

$$\langle x'\, y'|x\, y\rangle = \langle 0|\psi(y')\psi(x')\psi^{+}(y)\psi^{+}(x)|0\rangle$$

$$\boxed{\psi(x)|0\rangle = 0}$$

$$= \langle 0|\psi(y')\left(\psi^{+}(y)\psi(x') + \delta(x'-y)\right)\psi^{+}(x)|0\rangle$$

$$= \delta(x'-y)\langle 0|\psi(y')\psi^{+}(x)|0\rangle$$

$$+ \langle 0|\psi(y')\psi^{+}(y)\psi(x')\psi^{+}(x)|0\rangle$$

$$= \delta(x'-y)\langle 0|\psi^{+}(x)\psi(y') + \delta(x-y')|0\rangle$$

$$+ \langle 0|\left(\psi^{+}(y)\psi(y') + \delta(y-y')\right)\left(\psi^{+}(x)\psi(x) + \delta(x-x')\right)|0\rangle$$

$$= \delta(x'-y)\delta(x-y') + \delta(y-y')\delta(x-x')$$

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3.3 Etats antisymétrisés (fermions)

$n=2 \quad |x\,y\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle|y\rangle - |y\rangle|x\rangle\right)$

$$|x\,y\rangle = -|y\,x\rangle$$

$$\langle x'y'|xy\rangle = \delta(x-x')\delta(y-y') - \delta(x-y')\delta(y-x')$$

$$\frac{1}{2}\int dx\,dy\; |x\,y\rangle\langle x\,y| = 1$$

$$\varphi(x,y) = \langle x\,y|\varphi\rangle$$

$$\varphi(x,y) = -\varphi(y,x)$$

Pour n particules

$$\boxed{|x_1\dots x_n\rangle = \frac{1}{\sqrt{n!}}\sum_{p\in S_n} \epsilon_p\, |x_{p(1)}\rangle\dots|x_{p(n)}\rangle}$$

$\epsilon_p =$ "signature" de la permutation

Remarque sur permutations

Permutations forment groupe de n éléments.

Composition

Permutation s'écrit comme un produit de transpositions,

Transposition :
"Echange" ou "permutation" de 2 des n objets

$$T_s\,(1\;2\;3\;\dots\;6)$$

$$=(1\;2\;6\;\dots\;3)$$

e.g. $(1\;2\;5\;3\;4\;\dots\;n)$

$$=(1\;2\;5\;4\;3\dots n)(1\;2\;4\;3\;5\dots n)$$

$1\;2\;3\;4\;5\;\dots\;n$

$1\;2\;4\;3\;5\;\dots\;n$

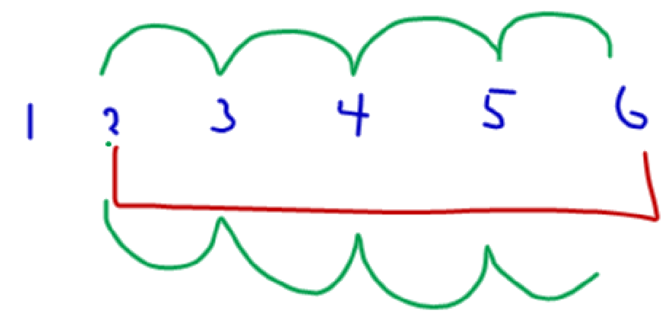
$1\;2\;5\;3\;4\;\dots\;n$

Permutation $\overset{\text{im}}{\text{paire}}$ $\left(\epsilon_p = \pm 1\right)$

si elle se décompose en

\# pair de transpositions

\# impair



$$\boxed{|x_{p(1)} \dots x_{p(n)}\rangle = \epsilon_p |x_1 \dots x_n\rangle}$$

$$\frac{1}{\sqrt{n!}} \sum_{q' \in S_n} \underset{p^{-1}q'}{\epsilon_q} |x_{\underset{q'}{q}p(1)}\rangle |x_{\underset{q'}{q}p(2)}\rangle \dots |x_{\underset{q'}{q}p(n)}\rangle$$

$$pq = q'$$

$$q = p^{-1} q'$$

$$\epsilon_{p^{-1}q'} = \epsilon_{p^{-1}} \epsilon_{q'}$$

$$\epsilon_p |x_1 \dots x_n\rangle \qquad \epsilon_p = \epsilon_{p^{-1}}$$

3.4 Création annihilation pour fermions

$$\psi^+(x)|x_1 \ldots x_n\rangle = |x\, x_1 \ldots x_n\rangle$$

Rajoute un fermion dans l'état propre de position x et antisymétrise avec les autres

$$\boxed{\psi^+(x)\psi^+(y) = -\psi^+(y)\psi^+(x)}$$

$$\{\psi^+(x), \psi^+(y)\} = 0$$

$$\{\psi(x), \psi(y)\} = 0$$

$$\{\psi(x), \psi^+(y)\} = \delta(x-y)$$

$$\psi^+(x)\psi(y)|\ldots\rangle$$

$$\psi(y)\psi^+(x)|\ldots\rangle$$

Exemple: $|x\,y\rangle = \psi^{+}(x)\,\psi^{+}(y)\,|0\rangle$

$$|y\,x\rangle = -|x\,y\rangle$$

$$\boxed{\psi(x)\,|0\rangle = 0}$$

$\langle x'\,y'\,|\,x\,y\rangle =$

$\langle 0|\,\psi(y')\,\psi(x')\,\psi^{+}(x)\,\psi^{+}(y)\,|0\rangle$

$= \langle 0|\,\psi(y')\left[-\psi^{+}(x)\,\psi(x') + \delta(x-x')\right]\psi^{+}(y)|0\rangle$

$= \delta(x-x')\,\langle 0|\,\underbrace{\psi(y')\,\psi^{+}(y)}_{-\psi^{+}(y)\cancel{\psi(y')}+\delta(y-y')}\,|0\rangle$

$(-\psi^{+}\psi + \delta(y'-x))\ (-\psi^{+}\psi + \delta(x'-y))$

$-\langle 0|\,[\psi(y')\,\psi^{+}(x)]\,[\psi(x')\,\psi^{+}(y)]\,|0\rangle$

$= \delta(x-x')\,\delta(y-y')$

$\quad - \delta(y'-x)\,\delta(x'-y)$

Transpositions

$\psi^{+}(x_1)\,\psi^{+}(x_2)\,\psi^{+}(x_3)\,\psi^{+}(x_4)\,\psi^{+}(x_5)\,\psi^{+}(x_6)\,|0\rangle$

$= -\psi^{+}(x_1)\,\psi^{+}(x_2)\,\psi^{+}(x_3)\,\psi^{+}(x_4)\,\psi^{+}(x_5)\,\psi^{+}(x_6)\,|$

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