Lectures: why

#1
Second quantization (30 min.)

Particle-Wave duality
convenient.

#2 Time ordered product, Green's function (45 min.)

Perturbation theory
Cross section → relation to Matsubara Green function

#3. Spectral weight, self-energy, quasiparticles (45 min.)

Relation to photoemission - Lehmann representation.
Analytic continuation.
Self-energy
Quasiparticles.

#4 Coherent state functional integral (90 min.)

Grassmann variables
Grassmann calculus.
Action: coherent state functional integral
Wick's theorem.

#5 Many-body perturbation theory (90 min.)

Source fields
Bethe-Salpeter — Dyson-Schwinger equation
Luttinger-Ward functional.

#6 Lindhard function, G.W. TPSC (90 min)

Hartree-Fock, RPA, GW, TPSC.

Lecture 1 (30 min.)

Main results from second quantization

87 Second quantization

87.1 Creation-annihilation operators

$$\{a_{\alpha_1}, a_{\alpha_2}\} = 0 \qquad \{a_{\alpha_1}, a^{+}_{\alpha_2}\} = \delta_{\alpha_1, \alpha_2}$$

Number operator

$$[n_{\alpha}, a^{+}_{\alpha}] = a^{+}_{\alpha} \qquad [n_{\alpha}, a_{\alpha}] = -a_{\alpha}$$

87.2 Change of basis

$$a^{+}_{\mu_m} = \sum_i a^{+}_{\alpha_i} \langle \alpha_i | \mu_m \rangle$$

87.2.1 Position - momentum basis

$$\psi^{+}(r)|0\rangle = |r\rangle \qquad c^{+}_{k}|0\rangle = |k\rangle$$

87.2.2 Wave-function

$$\langle r_1 \ldots r_N | \alpha_1 \ldots \alpha_N \rangle = \frac{1}{\sqrt{N!}} \det \begin{bmatrix} \varphi_{\alpha_1}(r_1) & \varphi_{\alpha_1}(r_2) & \ldots & \varphi_{\alpha_1}(r_N) \\ \vdots & & & \\ \varphi_{\alpha_N}(r_1) & \varphi_{\alpha_N}(r_2) & \ldots & \varphi_{\alpha_N}(r_N) \end{bmatrix}$$

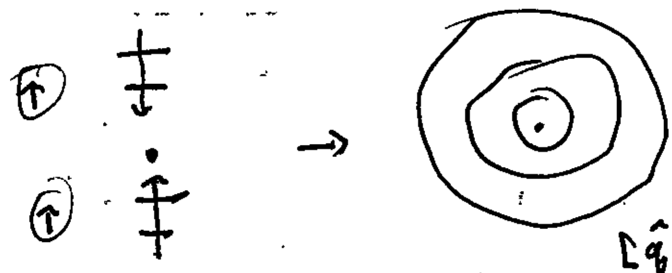
87.3 One-body operator

$$\hat{V} = \sum_{\sigma} \int d^3r \, V(r) \, \psi^{+}_{\sigma}(r) \psi_{\sigma}(r)$$

87.4 Two-body operator

$$\frac{1}{2} \sum_{\sigma\sigma'} \int d^3x \, d^3y \, v(x-y) \, \psi^{+}_{\sigma}(x) \psi^{+}_{\sigma'}(y) \psi_{\sigma'}(y) \psi_{\sigma}(x)$$

87. Second quantization



In the far field
Interference no: Pauli

$p = \partial L / \partial \dot{q}$

Creation annihilation: Why: $\frac{[\hat{q}, \hat{p}]}{i\hbar} = 1 \leftrightarrow \{q, p\}_{\text{Poisson}}$ particle $\rightarrow$ wave

$\langle \alpha_i | \alpha_j \rangle = \delta_{ij}$ α = basis e.g. $\vec{r}, \vec{k} \ldots$
i = which element

2 particles
$$|\alpha_1 \alpha_2\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_1\rangle |\alpha_2\rangle - |\alpha_2\rangle |\alpha_1\rangle \right) = -|\alpha_2 \alpha_1\rangle$$

Creation operator (Fock space)

$a^+_{\alpha_i} |0\rangle = |\alpha_i\rangle$ add + antisymmetrizes

$$|\alpha_1 \alpha_2\rangle = a^+_{\alpha_1} a^+_{\alpha_2} |0\rangle = -a^+_{\alpha_2} a^+_{\alpha_1} |0\rangle = -|\alpha_2 \alpha_1\rangle$$

$$\boxed{0 = \{a^+_{\alpha_1}, a^+_{\alpha_2}\}} = a^+_{\alpha_1} a^+_{\alpha_2} + a^+_{\alpha_2} a^+_{\alpha_1}$$

Initial order arbitrary.
Works if interchange any 2 in the list

Annihilation:

$$\langle \alpha_i | = \langle 0 | a_{\alpha_i} \Rightarrow \boxed{a_{\alpha_i} = (a^+_{\alpha_i})^+}$$

$$\langle \alpha_i | 0 \rangle = \langle 0 | a_{\alpha_i} | 0 \rangle = 0 \Rightarrow \boxed{a_{\alpha_i} |0\rangle = 0}$$

Last anticommutation:

$$\langle \alpha_i | \alpha_j \rangle = \langle 0 | a_{\alpha_i} a^+_{\alpha_j} | 0 \rangle = \delta_{ij}$$

$$\boxed{\{a_{\alpha_i}, a^+_{\alpha_j}\} = \delta_{ij}}$$

Number operator

$$\hat{n}_{\alpha_1} = a^{+}_{\alpha_1} a_{\alpha_1} \qquad \hat{n}_{\alpha_1}|0\rangle = 0$$

$$\hat{n}_{\alpha_i} a^{+}_{\alpha_j}|0\rangle = a^{+}_{\alpha_i} a_{\alpha_i} a^{+}_{\alpha_j}|0\rangle$$

$$= a^{+}_{\alpha_i}(\delta_{ij} - a^{+}_{\alpha_j} a_{\alpha_i})|0\rangle$$

$$= a^{+}_{\alpha_i}(\delta_{ij})|0\rangle$$

$$\boxed{[\hat{n}_{\alpha_i}, a^{+}_{\alpha_i}]} = a^{+}_{\alpha_i} a_{\alpha_i} a^{+}_{\alpha_i} - \cancel{a^{+}_{\alpha_i} a^{+}_{\alpha_i}}^{\to 0} a_{\alpha_i}$$

$$\downarrow$$

$$a^{+}_{\alpha_i}(1 - a^{+}_{\alpha_i} a_{\alpha_i}) = \boxed{a^{+}_{\alpha_i}}$$

$$\boxed{[\hat{n}_{\alpha_i}, a_{\alpha_i}] = -a_{\alpha_i}}$$

Change of basis

$$|\mu_m\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\mu_m\rangle$$

$$a^{+}_{\mu_m} = \sum_i a^{+}_{\alpha_i}\langle\alpha_i|\mu_m\rangle$$

$$\{a_{\mu_m}, a^{+}_{\mu_n}\} = \sum_{ij}\langle\mu_m|\alpha_i\rangle\{a_{\alpha_i}, a^{+}_{\alpha_j}\}\langle\alpha_j|\mu_n\rangle$$

$$= \delta_{\mu_m \mu_n}$$

Wave function

$$\hat{\Psi}(r) = \sum_i \langle r|\alpha_i\rangle a_{\alpha_i} = \sum_i \varphi_i(r)\, a_{\alpha_i}$$

$$\frac{1}{\sqrt{2}}\langle r_1, r_2|\alpha_1, \alpha_2\rangle = \Psi_{\alpha_1,\alpha_2}(r_1, r_2)$$

$$= \frac{1}{\sqrt{2}}\langle 0|\hat{\Psi}(r_2)\hat{\Psi}(r_1)\, a^{+}_{\alpha_1} a^{+}_{\alpha_2}|0\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_i \sum_j \varphi_i(r_2)\varphi_j(r_1) \langle 0 | a_{\alpha_i} a_{\alpha_j} a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} | 0 \rangle$$

$$\delta_{1j}\delta_{2i} - \delta_{2j}\delta_{1i}$$

$$= \frac{1}{\sqrt{2}} \left(\varphi_2(r_2)\varphi_1(r_1) - \varphi_1(r_2)\varphi_2(r_1) \right)$$

$$= \frac{1}{\sqrt{2}} \det \begin{bmatrix} \varphi_1(r_1) & \varphi_1(r_2) \\ \varphi_2(r_1) & \varphi_2(r_2) \end{bmatrix}$$

<u>One-body operators</u>

$$\sum_{\ell=1}^{N=3} [V(\hat{R}_1) + V(\hat{R}_2) + V(\hat{R}_3)]\Psi$$

$\ell \to$ ptle #

Diagonal basis

$$\hat{U} |\alpha_i\rangle = U_{\alpha_i} |\alpha_i\rangle = \langle \alpha_i | \hat{U} | \alpha_i \rangle |\alpha_i\rangle$$

e.g. $V(\hat{R})|r\rangle = V(r)|r\rangle$

In general $\forall$ # of particles.

$$\sum_i U_{\alpha_i} \hat{n}_{\alpha_i} = \sum_{ij} a^\dagger_{\alpha_i} \underbrace{\langle \alpha_i | \hat{U} | \alpha_j \rangle}_{\delta_{ij}} a_{\alpha_j}$$

Change of basis:

$$= \sum_{m,n} a^\dagger_{\mu_m} \langle \mu_m | \hat{U} | \mu_n \rangle a_{\mu_n}$$

e.g. $\boxed{\hat{V} = \sum_\sigma \int d^3r \, V(r) \psi^\dagger_\sigma(r) \psi_\sigma(r)}$

<u>Two-body</u> Diagonal: $\frac{1}{2} \sum_{ij} \langle \alpha_i | \langle \alpha_j | \hat{V} | \alpha_i \rangle | \alpha_j \rangle$

$$\hat{n}_{\alpha_i} \hat{n}_{\alpha_j} - \delta_{ij} \hat{n}_{\alpha_i}$$

$$= \frac{1}{2} \sum_{ij} (\alpha_i \alpha_j | V | \alpha_i \alpha_j) a^\dagger_{\alpha_i} a^\dagger_{\alpha_j} a_{\alpha_j} a_{\alpha_i}$$

= result in summary

Lecture #2

Time-ordered product, Green's function

83. Perturbation theory

$$e^{-\beta \hat{K}} = e^{-\beta K_0} \hat{U}(\beta) \qquad \hat{U}(\beta) = T_\tau \left[e^{-\int_0^\beta \hat{K}_1(\tau)\, d\tau} \right]$$

$$\hat{K}_1(\tau) = e^{\hat{K}_0 \tau} \hat{K}_1 e^{-\hat{K}_0 \tau}$$

84.1 Photo-emission

$$\frac{\partial^2 \sigma}{\partial \Omega \partial \omega} \propto \sum_{m,n} e^{-\beta K_m} \langle m | c_k^\dagger | n \rangle \langle n | c_k | m \rangle \, \delta(\omega - (K_m - K_n))$$

29.1 Matsubara Green function

$$\mathcal{G}_{\alpha\beta}(\tau) = -\langle T_\tau c_\alpha(\tau) c_\beta^\dagger \rangle$$

$$= -\langle c_\alpha(\tau) c_\beta^\dagger \rangle \theta(\tau) + \langle c_\beta^\dagger(0) c_\alpha(\tau) \rangle \theta(-\tau)$$

29.3 Antiperiodicity + Fourier

$$\mathcal{G}_{\alpha\beta}(ik_n) = \int_0^\beta d\tau\, e^{ik_n \tau} \mathcal{G}_{\alpha\beta}(\tau)$$

$$\mathcal{G}_{\alpha\beta}(\tau) = T \sum_n e^{-ik_n \tau} \mathcal{G}_{\alpha\beta}(ik_n)$$

29.8 Non-interacting

$$\mathcal{G}_k(ik_n) = \frac{1}{ik_n - \xi_k}$$

29.2 Time-order in practice

$$\langle T_\tau \psi(\tau_1) \psi^\dagger(\tau_3) \psi(\tau_2) \psi^\dagger(\tau_4) \rangle$$

$$= -\langle T_\tau \psi^\dagger(\tau_3) \psi(\tau_1) \psi(\tau_2) \psi^\dagger(\tau_4) \rangle$$

Perturbation theory

$$\hat{K} = \hat{H} - \mu \hat{N} \qquad \hat{K} = \hat{K}_0 + \hat{K}_1 \qquad [K_0, K_1] \neq 0$$

$$e^{-\beta(\hat{K}_0 + \hat{K}_1)} \neq e^{-\beta \hat{K}_0} e^{-\beta \hat{K}_1} \quad \text{(power series)}$$

$$\boxed{Z = \mathrm{Tr}\left[e^{-\beta \hat{K}}\right]}$$

$$e^{-\beta \hat{K}} = e^{-\beta \hat{K}_0} \hat{U}(\beta)$$

$$\frac{\partial}{\partial \tau} e^{-\tau \hat{K}} = -\hat{K} e^{-\tau \hat{K}} = -\cancel{(\hat{K}_0 + \hat{K}_1)} e^{-\tau \hat{K}_0} \hat{U}(\tau)$$

$$= -\hat{K}_0 \cancel{e^{-\tau \hat{K}_0}} \hat{U}(\tau) + e^{-\tau \hat{K}_0} \frac{\partial \hat{U}(\tau)}{\partial \tau}$$

$$\frac{\partial \hat{U}(\tau)}{\partial \tau} = -e^{\tau \hat{K}_0} \hat{K}_1 e^{-\tau \hat{K}_0} \hat{U}(\tau) = -\hat{K}_1(\tau) U(\tau)$$

$$\boxed{\hat{K}_1(\tau) = e^{+\tau \hat{K}_0} \hat{K}_1 e^{-\tau \hat{K}_0}}$$

$$\hat{U}(\beta) - 1 = -\int_0^\beta d\tau \, \hat{K}_1(\tau) \hat{U}(\tau)$$

$$\hat{U}(\beta) = 1 - \int_0^\beta d\tau \, \hat{K}_1(\tau) + \int_0^\beta d\tau \, K_1(\tau) \int_0^\tau d\tau' K_1(\tau')$$

$$\underbrace{- \int_0^\beta d\tau \, \hat{K}_1(\tau) \int_0^\tau d\tau' \hat{K}_1(\tau') \int_0^{\tau'} d\tau'' \hat{K}_1(\tau'')}_{- \frac{1}{3!} T_\tau \int_0^\beta d\tau \, \hat{K}_1(\tau)}$$

$$\boxed{\hat{U}(\beta) = T_\tau \left[e^{-\int_0^\beta d\tau \, \hat{K}_1(\tau)} \right]}$$

$\mathcal{G}$ Matsubara

Definition:

T_τ motivated by perturbation theory

$$\mathcal{G}_{\alpha\beta}(\tau) = -\langle T_\tau\, c_\alpha(\tau)\, c_\beta^+\rangle$$

$$= -\langle c_\alpha(\tau)\, c_\beta^+\rangle\,\theta(\tau) + \langle c_\beta^+\, c_\alpha(\tau)\rangle\,\theta(-\tau)$$

$$c_\alpha(\tau) = e^{K\tau}\, c_\alpha\, e^{-K\tau} \qquad c_\beta^+(\tau) = e^{K\tau}\, c_\beta^+\, e^{-K\tau}$$

$\uparrow$ not the adjoint

$$\boxed{\tau = it/\hbar}$$

Antiperiodicity

$$\tau > 0 \Rightarrow \mathcal{G}_{\alpha\beta}(\tau) = -\mathcal{G}_{\alpha\beta}(\tau-\beta)$$

$$\mathcal{G}_{\alpha\beta}(\tau) = -\frac{1}{Z}\,\mathrm{Tr}\left[e^{-\beta\hat{K}}\, e^{\hat{K}\tau}\, c_\alpha\, e^{-\hat{K}\tau}\, c_\beta^+\right]$$

$$= -\frac{1}{Z}\,\mathrm{Tr}\left[\left(e^{\beta\hat{K}}\, e^{-\beta\hat{K}}\right) c_\beta^+\, e^{-\beta\hat{K}}\, e^{\hat{K}\tau}\, c_\alpha\, e^{-\hat{K}\tau}\right]$$

$$= -\frac{1}{Z}\,\mathrm{Tr}\left[e^{-\beta\hat{K}}\, c_\beta^+\, c_\alpha(\tau-\beta)\right]$$

$$= -\mathcal{G}_{\alpha\beta}(\tau-\beta)$$

Fourier

$$\boxed{\mathcal{G}_{\alpha\beta}(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-ik_n\tau}\,\mathcal{G}_{\alpha\beta}(ik_n)\ ;\quad \mathcal{G}_{\alpha\beta}(ik_n) = \int_0^\beta d\tau\, e^{ik_n\tau}\,\mathcal{G}_{\alpha\beta}(\tau)}$$

$$k_n = (2n+1)\pi T \quad (k_B = 1)$$

Non-interacting, diagonal basis

$$\hat{K}_0 = \sum_k \xi_k c_k^+ c_k = \sum_k \xi_k \hat{n}_k$$

$$\frac{\partial c_k(\tau)}{\partial \tau} = [\hat{K}_0, c_k] = -\xi_k c_k(\tau)$$

$$\mathcal{G}_k(\tau > 0) = -\langle c_k(\tau) c_k^+ \rangle = -e^{-\xi_k \tau} \langle c_k c_k^+ \rangle$$

$$= -e^{-\xi_k \tau}(1 - f(\xi_k))$$

$$\mathcal{G}_k(ik_n) = -\int_0^\beta d\tau\, e^{ik_n\tau}\left(e^{-\xi_k\tau}(1-f(\xi_k))\right)$$

$$= -\frac{e^{(ik_n - \xi_k)\tau}}{ik_n - \xi_k}\Bigg|_0^\beta \frac{e^{+\beta\xi_k}}{1+e^{+\beta\xi_k}}$$

$$= +\frac{(e^{-\beta\xi_k}+1)}{ik_n - \xi_k}\,\frac{1}{e^{-\beta\xi_k}+1} = \frac{1}{ik_n - \xi_k}$$

N.B. Sum over Matsubara Need for Lindhard

$$\frac{\partial \mathcal{G}_k(\tau)}{\partial \tau} = -\delta(\tau)\langle \{c_k(\tau), c_k^+\}\rangle - \langle T_\tau \frac{\partial c_k(\tau)}{\partial \tau} c_k^+ \rangle$$

$$= -\delta(\tau) - \xi_k \mathcal{G}_k(\tau)$$

$$\int_0^\beta d\tau\, e^{ik_n\tau}\left(\frac{\partial}{\partial\tau} + \xi_k\right)\mathcal{G}_k(\tau) = -1$$ ← Structure

$$\underline{e^{ik_n\tau}\mathcal{G}_k(\tau)\Big|_0^\beta} - ik_n \mathcal{G}_k(ik_n) = -1 - \xi_k \mathcal{G}_k(ik_n)$$

$$\mathcal{G}_k(\beta) + \mathcal{G}_k(0) = 0$$

Lecture #3 (45 min)

84.4 Spectral weight and how it is related to $\mathcal{G}_k(ik_n)$ and to photoemission

$$\frac{\partial^2\sigma}{\partial\Omega\,\partial\omega} \propto A_k(\omega)\, f(\omega)$$

29.5 Lehmann representation

$$\mathcal{G}_k(ik_n) = \int \frac{d\omega}{2\pi} \frac{A_k(\omega)}{ik_n - \omega}$$

⤳ Significance of poles

$$A_k(\omega) = \sum_{m,n} \frac{1}{Z}\left(e^{-\beta K_n} + e^{-\beta K_m}\right) \langle n|c_k|m\rangle \langle m|c_k^{+}|n\rangle$$

$$2\pi\, \delta(\omega - (K_m - K_n))$$

29.6 Spectral weight from $\mathcal{G}_k(ik_n)$ analytic continuation

$$G_k^R(\omega) = \int \frac{d\omega'}{2\pi} \frac{A_k(\omega')}{\omega + i\eta - \omega'}$$

Ch. 17. Self-energy.

$$A_k(\omega) = \frac{2\Gamma}{(\omega - \tilde{\xi}_k)^2 + \Gamma^2} \Rightarrow G_k^R(\omega) = \frac{1}{\omega - \tilde{\xi}_k + i\Gamma}$$

$$G_k^R(\omega) = \frac{1}{\omega + i\eta - \xi_k - \Sigma_k^R(\omega)}\ ; \quad G_k^{R^{-1}}(\omega) = G_k^{0\,R^{-1}}(\omega) - \Sigma_k^R(\omega)$$

$$G_k^R(t) = -i\,\theta(t)\, e^{-i\tilde{\xi}_k t - \Gamma t}\ ; \quad |\langle k|\psi(t)\rangle|^2 = \theta(t)\, e^{-2\Gamma t}$$

18.3 Poles

85.3 $\mathrm{Im}\,\Sigma_k^R(\omega) < 0$

31.3 Experiments

31.4 Quasiparticles

31.5 Fermi liquid.

Lehmann representation

$$\mathcal{G}_k(ik_n) = -\int_0^\beta d\tau \sum_{nm} e^{-\beta K_n} \langle n| e^{K_n \tau} c_k e^{-K_m \tau} |m\rangle\langle m| c_k^\dagger |n\rangle \quad \left(e^{ik_n\tau} \right)$$

$$= -\sum_{nm} e^{-\beta K_n} \frac{e^{(ik_n - (K_m - K_n))\tau}}{ik_n - (K_m - K_n)} \Bigg|_0^\beta |\langle n| c_k |m\rangle|^2$$

$$= \sum_{nm} \frac{(e^{-\beta K_n} + e^{-\beta K_m}) |\langle n| c_k |m\rangle|^2}{ik_n - (K_m - K_n)} \rightsquigarrow \text{Significance of poles}$$

$$= \int \frac{d\omega}{2\pi} \frac{A_k(\omega)}{ik_n - \omega}$$

where

$$A_k(\omega) = \sum_{n,m} e^{-\beta K_n} \langle n| c_k |m\rangle\langle m| c_k^\dagger |n\rangle 2\pi\, \delta(\omega - (K_m - K_n)) \, (1 + e^{\beta\omega})$$

$$\Rightarrow \frac{\partial^2 \sigma}{\partial\Omega\, \partial\omega} \propto A_k(\omega) f(\omega)$$

Analytic continuation

$$G_k^R(\omega) = \int \frac{d\omega'}{2\pi} \frac{A_k(\omega')}{\omega + i\eta - \omega'} \qquad \lim_{\eta\to 0} \frac{1}{\omega + i\eta - x} = \mathcal{P}\left(\frac{1}{\omega - x}\right) - i\pi\delta(\omega - x)$$

$$A_k(\omega) = -2\, \mathrm{Im}\, G^R(\omega)$$

Self-energy

$$A_k(\omega) = -2\,\mathrm{Im}\,\frac{1}{\omega + i\eta - \xi_k} = 2\pi\,\delta(\omega - \xi_k)$$

Life-time $\quad A_k(\omega) = \dfrac{+2\Gamma}{(\omega - \tilde{\xi}_k)^2 + \Gamma^2}$

$$G^R_k(\omega) = \int \frac{d\omega'}{2\pi}\left(\frac{1}{(\omega' - \tilde{\xi}_k) - i\Gamma} - \frac{1}{(\omega' - \tilde{\xi}_k) + i\Gamma}\right)\frac{1}{i}\,\frac{1}{\omega + i\eta - \omega'}$$

$$= \frac{(-1)^2}{\omega - \tilde{\xi}_k + i\Gamma} \equiv \frac{1}{\omega + i\eta - \xi_k - \Sigma^R_k(\omega)}$$

$$\Sigma^R_k(\omega) = \tilde{\xi}_k - \xi_k - i\Gamma$$

N.B. $\boxed{\mathrm{Im}\,\Sigma^R(\omega) < 0 \text{ for causality.}}$

Define $\boxed{G^{R\,-1}_k(\omega) = G^{0R\,-1}_k(\omega) - \Sigma_k(\omega)}$

$$\boxed{\mathcal{G}^{-1}_k(ik_n) = \mathcal{G}^{0\,-1}_k(ik_n) - \Sigma_k(ik_n)}$$

$$G^R_k(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i(\omega + i\eta)t}\,\frac{1}{\omega - \tilde{\xi}_k + i\Gamma} = \frac{-2\pi i}{2\pi}\,e^{-i(\tilde{\xi}_k - i\Gamma)t}\,\theta(t)$$

$$= -i\,\theta(t)\,e^{-i\tilde{\xi}_k t - \Gamma t} \quad ; \quad |\langle k|\psi_k(t)\rangle|^2 = \theta(t)\,e^{-2\Gamma t}$$

N.B. $-i(\omega_1 + i\omega_2)t$

$\to -i\omega_1 + \omega_2 t$

$\Rightarrow$ if $t < 0$

$\omega_2 > 0$

no pole

$\mathcal{G}$ is reducible

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0\Sigma\mathcal{G}_0 + \mathcal{G}_0\Sigma\mathcal{G}_0\Sigma\mathcal{G}_0 + \cdots$$

Quasiparticles

$$A_k(\omega) = \frac{-2\Sigma''_k(\omega)}{(\omega - \xi_k - \Sigma'_k(\omega))^2 + (\Sigma'')^2}$$

$$Z_k^{-1} = 1 - \frac{\partial \Sigma'_k}{\partial \omega} \Rightarrow A_k(\omega) = \frac{-2\Sigma''_k(\omega)}{(Z^{-1}\omega - \xi_k)^2 + \Sigma''^2_k}$$

$$A_k(\omega) = Z_k \frac{(-2Z\Sigma''_k(\omega))}{(\omega - Z_k\xi_k)^2 + (Z\Sigma''_k(\omega))^2} + (\text{Inc.})$$

Inc. necessary because

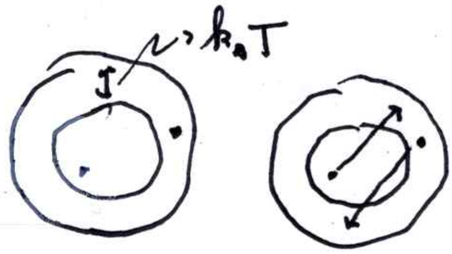
$$\int \frac{d\omega}{2\pi} A_k(\omega) = \langle \{c_k, c_k^\dagger\} \rangle = 1$$

$\uparrow$

cf. Lehmann.

Fermi liquid

$$\Sigma''_k(\omega) = -a(\omega^2 + (\pi T)^2)$$



Lecture #4

79. Coherent states for fermions

$c|\eta\rangle = \eta|\eta\rangle \qquad (\alpha\eta)^\dagger = \alpha^*\eta^+$

$\{\eta_1, \eta_2\} = \{\eta_1, \eta_2^+\} = 0 \qquad F(\eta) = a + b\eta$

Second quatization $|\eta\rangle = (1 - \eta c^+)|0\rangle = e^{-\eta c^+}|0\rangle$

$\langle\eta| = \langle 0|(1 - c\eta^+)|0\rangle$

Calculus $\int d\eta = 0 \quad \int d\eta\,\eta = 1\ ; \quad \int d\eta \frac{\partial f}{\partial\eta} = 0$

Linearity $\eta^+ \frac{\partial}{\partial\eta} = -\frac{\partial}{\partial\eta}\eta^+$

Change of variables $(x = y/a)$

$$\int d\eta\, F(\eta) = \int d\eta\,(a + b\eta) = b$$

$$d\eta = \frac{d\eta'}{a} \Rightarrow \int \frac{d\eta'}{a}\left(a + b\frac{\eta'}{a}\right) = \frac{b}{a^2} \Rightarrow d\eta = a\, d\eta'$$

Many variables $\quad \psi_i = \sum_j U_{ij}\eta_j$

$$\prod_i \int d\psi_i = \sum_{j_1=1}^{N} \cdots \sum_{j_N=1}^{N} U_{ij_1} U_{ij_2} \cdots U_{ij_N} \int d\eta_{j_1} \cdots d\eta_{j_n}$$

$$= \det(U) \int d\eta_1 \cdots d\eta_N$$

Gaussian integral $\int d\eta^+ d\eta\, e^{-\eta^+ a\eta} = \int d\eta^+ d\eta\,(1 - \eta^+ a\eta) = a$

$$\int \mathcal{D}\eta^+ \int \mathcal{D}\eta = \prod_i \int d\eta_i^+ d\eta_i$$

$$\boxed{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta\, e^{-\eta^+ A\eta} = \det A = \exp(\mathrm{Tr}\ln A)}$$

$$\int \mathcal{D}\eta^+ \int \mathcal{D}\eta \, e^{-\eta^+ A \eta - \eta^+ J - J^+ \eta} = \det A \, e^{J^+ A^{-1} J}$$

Closure, over complete, Trace:

$$\int d\eta^+ \int d\eta \, e^{-\eta^+ \eta} |\eta\rangle\langle\eta| = 1$$

Over complete:

$$\langle \eta_1 | \eta_2 \rangle = e^{\eta_1^+ \eta_2}$$

Trace:

$$\mathrm{Tr}[O] = \int d\eta^+ \int d\eta \, e^{-\eta^+ \eta} \langle -\eta | O | \eta \rangle$$

80. <u>Coherent state functional integral</u> (1 fermion)

$$Z = \int \mathcal{D}\eta^+ \int \mathcal{D}\eta \, e^{-S} \qquad S = \int_0^\beta d\tau \left(\eta^+(\tau) \frac{\partial}{\partial \tau} \eta(\tau) + H[\eta^+(\tau), \eta(\tau)] \right)$$

$$S^0 = + \sum_n \eta^+(ik_n) \left(-\mathcal{G}^{0^{-1}}(ik_n) \right) \eta(ik_n)$$

$$Z = \exp \sum_n \ln \left(-\mathcal{G}^{-1}(ik_n) \right) e^{-ik_n 0^+}$$

$$\mathcal{G} = - \frac{\int d\eta^+ d\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta} \, \eta \eta^+}{\int d\eta^+ d\eta \, e^{-\eta(-\mathcal{G}^{-1})\eta}} = \frac{-1}{-\mathcal{G}^{-1}} = \mathcal{G}$$

<u>Wick's theorem</u>

$$(-1)^m \frac{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta} \, \eta_1 \eta_1^+ \eta_2 \eta_2^+ \cdots \eta_m \eta_m^+}{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta}$$

$$= \mathcal{G}_{11} \mathcal{G}_{22} \cdots \mathcal{G}_{mm} = \det(\mathcal{G})$$

4.0c

$$(-1)^m \langle c(\tau_m)\, c^+(\tau_{m'}) \ldots c(\tau_2)\, c^+(\tau_{2'})\, c(\tau_1)\, c^+(\tau_1') \rangle$$

$$= \det \begin{bmatrix} \mathcal{G}(\tau_1, \tau_1') & \mathcal{G}(\tau_1, \tau_2') & \ldots & \mathcal{G}(\tau_1, \tau_m') \\ \mathcal{G}(\tau_2, \tau_1') & \mathcal{G}(\tau_2, \tau_2') & \ldots & \mathcal{G}(\tau_2, \tau_m') \\ \vdots & & & \\ \mathcal{G}(\tau_m, \tau_1') & \mathcal{G}(\tau_m, \tau_2') & \ldots & \mathcal{G}(\tau_m, \tau_m') \end{bmatrix}$$

Lecture #4 (90 min)

79. Coherent states for fermions

79.1 Grassmann variables for fermions

$$c|\eta\rangle = \eta|\eta\rangle \quad ; \quad c_1 c_2 |\eta_1 \eta_2\rangle = -c_2 c_1 |\eta_1 \eta_2\rangle$$

$$\Rightarrow \boxed{\{\eta_1, \eta_2\} = 0}$$

$$(\alpha\eta)^+ = \alpha^* \eta^+ \qquad \{\eta, \eta^+\} = 0$$

Link to second quantization:

$$|\eta\rangle = (1 - \eta c^+)|0\rangle$$

$$c|\eta\rangle = \eta c c^+ |0\rangle = \eta|0\rangle = \eta(1 - \eta c^+)|0\rangle = \eta|\eta\rangle$$

Other representation:

$$|\eta\rangle = e^{-\eta c^+}|0\rangle$$

Adjoint: (exercise)

$$\langle\eta| = \langle 0|(1 - c\eta^+)$$

$$\langle\eta|c^+ = \langle 0|(c^+ - c\eta^+ c^+) = \langle 0|c c^+ \eta^+$$

$$= \langle 0|\eta^+ = \langle 0|(1 - c\eta^+)\eta^+ = \langle\eta|\eta^+$$

79.2 Grassmann calculus

Integrals as if $-\infty$ to ∞

$$\int d\eta\, f(\eta + \xi) = \int d\eta\, f(\eta) \; ; \quad f(\eta) = a + b\eta$$

$$\Rightarrow \boxed{\begin{array}{l} \int d\eta = 0 \; ; \; \int d\eta\, \eta = 1 \\ \frac{\partial a}{\partial \eta} = 0 \; ; \; \frac{\partial b\eta}{\partial \eta} = b \end{array}} \Rightarrow \boxed{\int d\eta \frac{\partial f}{\partial \eta} = 0}$$

<u>Linearity</u> $\int d\eta\,(a f(\eta) + b g(\eta)) = a\int d\eta\, f(\eta) + b\int d\eta\, g(\eta)$

Anticomm. $\boxed{\eta^{+}\frac{\partial}{\partial\eta} = -\frac{\partial}{\partial\eta}\eta^{+}}$

<u>Delta:</u> $\int d\eta\,\delta(\eta'-\eta)\,F(\eta) = \int d\eta\,(\eta-\eta')\,F(\eta) = F(\eta')$

79.3 Change of variables

Ordinary variables: $\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi}$

$x = y/a \Rightarrow \underbrace{\frac{1}{a}}_{\text{Jacobian.}}\int dy\, e^{-y^2/2a^2} = \frac{1}{a}\sqrt{2\pi a^2}$

Grassmann:

$\int d\eta\, F(\eta) = \int d\eta\,(a + b\eta) = b$

$d\eta = \frac{d\eta'}{a} \Rightarrow \int \frac{d\eta'}{a}\left(a + b\frac{\eta'}{a}\right) = \frac{b}{a^2}$

So, we must use for Jacobian $\boxed{d\eta = a\, d\eta'}$

The inverse of what we normally do

Plusieurs variables: $\psi_i = \sum_j U_{ij} \eta_j$

$$\prod_{i=1}^{N} \int d\psi_i = \prod_{i=1}^{N} \sum_{j_i} U_{ij_i} \int d\eta_{j_i}$$

All j_i different

$$= \sum_{j_1=1}^{N} \cdots \sum_{j_N=1}^{N} U_{1j_1} \cdots U_{Nj_N} \int d\eta_{j_1} \cdots d\eta_{j_N}$$

in order $\Rightarrow$

$$= \sum_{j_1=1}^{N} \cdots \sum_{j_N=1}^{N} U_{1j_1} \cdots U_{Nj_N} \epsilon^{j_1 j_2 \cdots j_N} \int d\eta_1 \cdots d\eta_N$$

$$= \det[U] \int d\eta_1 \cdots d\eta_N$$

Same reasoning for integral over $f(\eta_1 \cdots \eta_N)$

$\Rightarrow$ Jacobian: $[\det U]^{-1}$

79.4 Grassmann Gaussian integrals

$$\int d\eta^+ \int d\eta\, e^{-\eta^+ a \eta} = \int d\eta^+ d\eta\, (1 - \eta^+ a \eta) = a$$

$$\int d\eta_1^+ \int d\eta_1\, e^{-\eta_1^+ a_1 \eta_1} \int d\eta_2^+ d\eta_2\, e^{-\eta_2^+ a_2 \eta_2} = a_1 a_2$$

since: expand $e^{-\eta_1^+ a_1 \eta - \eta_2^+ a_2 \eta} = 1 - \eta_1^+ a_1 \eta_1 - \eta_2^+ a_2 \eta_2 + \eta_1^+ a \eta_1 \eta_2^+ a_2 \eta_2$

$$= e^{-\eta_1^+ a_1 \eta_1} e^{-\eta_2^+ a_2 \eta_2}$$

$\Rightarrow$ $\boxed{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta\, e^{-\eta^+ A \eta} = \det[A] = \exp[\mathrm{Tr} \ln A]}$

$\prod_i \int d\eta_i^+ d\eta_i$

Source fields:

$$\int d\eta^+ d\eta \; e^{-\eta^+ a \eta - \eta^+ J - J^+ \eta}$$

$$= \int d\eta^+ \int d\eta \; e^{-(\eta^+ + J^+ a^{-1}) a (\eta + a^{-1} J) + J^+ a^{-1} J}$$

$$= a\, e^{J^+ a^{-1} J} \Rightarrow \text{in general } \boxed{\det A \; e^{+J^+ A^{-1} J}}$$

79.5 Closure, overcompleteness, Trace formula

Closure $\int d\eta^+ \int d\eta \; e^{-\eta^+ \eta} \, |\eta\rangle\langle\eta| = \int d\eta^+ \int d\eta \, (1 - \eta^+ \eta)(1 - \eta c^+)|0\rangle$ ①

① $\langle 0|(1 - c\eta^+)$ ②

$$= |0\rangle\langle 0| + \int d\eta^+ d\eta \, (-\eta|1\rangle)(\langle 1|(-\eta^+))$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

Overcompleteness

$$\langle \eta_1 | \eta_2 \rangle = \langle 0|(1 - c\eta_1^+)(1 - \eta_2 c^+|0\rangle$$

$$= 1 + \langle 1|\eta_1^+ \eta_2|1\rangle = 1 + \eta_1^+ \eta_2 = e^{\eta_1^+ \eta_2}$$

Trace

$$\mathrm{Tr}[O] = \int d\eta^+ \int d\eta \; e^{-\eta^+ \eta} \langle -\eta | O | \eta \rangle$$

$$= \int d\eta^+ \int d\eta \, (1 - \eta^+ \eta) \langle 0|(1 + c\eta^+) O (1 - \eta c^+)|0\rangle$$ ① ②

$$= \langle 0|O|0\rangle + \langle 1|O|1\rangle$$

80, Coherent state functional integral for fermions

80.1 Simple example with single fermion

Trotter $e^{-\beta(\hat{T}+\hat{V})} = \prod_{i=1}^{N_\tau} e^{-\Delta\tau\hat{T}} e^{-\Delta\tau\hat{V}}$

Closure $\int d\eta^+ d\eta\, e^{-\eta^+\eta} |\eta\rangle\langle\eta|$

Trace $\int d\eta_0^+ d\eta_0\, e^{-\eta_0^+\eta_0} \langle -\eta_0| \quad |\eta_0\rangle$

$$\boxed{Z = \int \mathcal{D}\eta^+ \mathcal{D}\eta\, e^{-S}}$$

$$\boxed{S = \int_0^\beta d\tau \left(\eta^+(\tau)\frac{\partial}{\partial\tau}\eta(\tau) + \hat{H}(\eta^+,\eta)\right)}$$

N.B. $\eta^+ = -\frac{\partial L}{\partial\dot{\eta}} \longleftrightarrow p = \frac{\partial L}{\partial\dot{q}}$ $\quad \boxed{L = p\dot{q} - H}$

$p\dot{q} = -\eta^+\dot{\eta}$ $\quad S = -\int L\,dt$

$$\int d\eta_2^+ \int d\eta_2\, e^{-\eta_2^+\eta_2} \langle\eta_2| e^{+H[c^+,c]\Delta\tau} |\eta_1\rangle$$

can be functions of τ

$$= \int d\eta_2^+ \int d\eta_2\, e^{-\eta_2^+\eta_2} \langle\eta_2|\eta_1\rangle e^{+\Delta\tau H[\eta_2^+,\eta_1]}$$

$$= \int d\eta_2^+ \int d\eta_2\, e^{-\eta_2^+\eta_2 + \eta_2^+\eta_1} e^{+\Delta\tau H[\eta_2^+,\eta_2]}$$

déjà $\Theta(\Delta\tau)$

$$= \int d\eta_2^+ \int d\eta_2\, e^{-\eta_2^+\left(\frac{\eta_2-\eta_1}{\Delta\tau}\right)\Delta\tau + H[\eta_2^+,\eta_2]\Delta\tau}$$

Diagonal basis if H indep. of τ:

$$\eta(\tau) = \sqrt{T} \sum_n e^{-i\omega_n \tau} \eta(i\omega_n)$$

Take ik_n

$$\eta^+(\tau) = \sqrt{T} \sum_n e^{i\omega_n \tau} \eta^+(i\omega_n)$$

$$\int_0^\beta d\tau \, \eta^+(\tau) \frac{\partial}{\partial \tau} \eta(\tau) = T \int_0^\beta d\tau \sum_n \sum_{n'} (-i\omega_n) e^{-i(\omega_n - \omega_n')\tau} \eta^+(i\omega_n') \eta(i\omega_n)$$

$$= \sum_n (-i\omega_n) \eta^+(i\omega_n) \eta(i\omega_n)$$

So $Z = \det\left(\frac{\partial}{\partial\tau} + H(\tau)\right) = \exp\left[\operatorname{Tr}\ln\left(\frac{\partial}{\partial\tau} + \epsilon\right)\right]$

take $H = \text{cte}$

$$= \exp \sum_n \ln(-i\omega_n + \epsilon) e^{-i\omega_n 0}$$

$$= \exp \sum_n \ln(-\mathcal{G}^{-1}(i\omega_n)) e^{-i\omega_n 0}$$

$$\det U^+ \det U = \det U^+U = e^{\operatorname{Tr}\ln U^+U}$$

$$U^+U = \delta(\tau - \tau')$$

N.B.

$$\boxed{\mathcal{G} = -\frac{\int d\eta^+ d\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta} \eta\eta^+}{\int d\eta^+ d\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta}} = \frac{-1}{-\mathcal{G}^{-1}} = \mathcal{G}}$$

$$\Rightarrow \boxed{S_0 = \sum_{n=-\infty}^{\infty} \eta^+(-ik_n + \epsilon)\eta}$$

§0.3 Wick's theorem

$$(-1)^m \frac{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta} \, \eta_1\eta_1^+ \eta_2\eta_2^+ \cdots \eta_m\eta_m^+}{\int \mathcal{D}\eta^+ \int \mathcal{D}\eta \, e^{-\eta^+(-\mathcal{G}^{-1})\eta}}$$

$$= \mathcal{G}_{11}\mathcal{G}_{22}\mathcal{G}_{33}\cdots\mathcal{G}_{mm} = \det(\mathcal{G})$$

$$(-1)^m \langle c(\tau_m)c^+(\tau_m') \cdots c(\tau_2)c^+(\tau_2') c(\tau_1)c^+(\tau_1') \rangle$$

$$= \det \begin{bmatrix} \mathcal{G}(\tau_1,\tau_1') & \mathcal{G}(\tau_1,\tau_2') & \cdots & \mathcal{G}(\tau_1,\tau_m') \\ \mathcal{G}(\tau_2,\tau_1') & \mathcal{G}(\tau_2,\tau_2') & \cdots & \mathcal{G}(\tau_2,\tau_m') \\ \vdots & & & \\ \mathcal{G}(\tau_m,\tau_1') & \mathcal{G}(\tau_m,\tau_2') & \cdots & \mathcal{G}(\tau_m,\tau_m') \end{bmatrix}$$

$\Rightarrow$ perturbation theory, same structure lecture # 5

87 Source fields for many-body

$$Z[\varphi] = \int \mathcal{D}\psi^+ \int \mathcal{D}\psi \,\hat{\exp}\left[-S - \psi^+(\bar{1})\,\varphi(\bar{1},\bar{2})\,\psi(\bar{2})\right]$$

$$-\frac{\delta \ln Z[\varphi]}{\delta\varphi(2,1)} = -\langle T\psi(1)\psi^+(2)\rangle_\varphi = \mathcal{G}(1,2)$$

$$\frac{\delta \mathcal{G}(1,2)_\varphi}{\delta\varphi(3,4)} = -\langle \psi^+(2)\psi(1)\psi^+(3)\psi(4)\rangle_\varphi + \mathcal{G}(1,2)_\varphi\,\mathcal{G}(4,3)_\varphi$$

Schwinger-Dyson

$$\left[\mathcal{G}_0^{-1}(1,\bar{2}) - \varphi(1,\bar{2})\right]\mathcal{G}(\bar{2},2)_\varphi$$

$$= \delta(1-2) - V(1-\bar{2})\langle \psi^+(\bar{2})\psi(\bar{2})\psi(1)\psi^+(2)\rangle_\varphi$$

$$-\Sigma(1,\bar{2})_\varphi\,\mathcal{G}(\bar{2},2)_\varphi$$

36.3 Four-point function : picture

36.4 Self-energy = picture : picture (irreducibility)

72. Luttinger-Ward : Free-energy $F[\varphi] = -T\ln Z[\varphi]$

$$\frac{1}{T}\frac{\delta F[\varphi]}{\delta\varphi(1,2)} = \mathcal{G}(2,1)$$

$$\Omega[\mathcal{G}] = F[\varphi] - \mathrm{Tr}[\varphi\mathcal{G}] \quad \text{Kadanoff-Baym}$$

$$\frac{1}{T}\frac{\delta\Omega}{\delta\mathcal{G}(1,2)} = -\varphi(2,1)$$

76. Constraining field

$$\left.\frac{\partial\Omega_\lambda[\mathcal{G}]}{\partial\lambda}\right|_{\mathcal{G}} = \left.\frac{\partial F_\lambda[\varphi]}{\partial\lambda}\right|_{\varphi}$$

5.0b

$$\Omega_\lambda[\mathcal{G}] = \Omega_{\lambda=0} + \int_0^1 d\lambda\, \frac{1}{\lambda} \langle \lambda V \rangle_\lambda$$

$$= F[\varphi_0] - \mathrm{Tr}[\varphi_0 \mathcal{G}] + \Phi[\mathcal{G}]$$

$\uparrow$ Luttinger Ward.

$$\frac{1}{T}\frac{\delta \Omega}{\delta \mathcal{G}} = -\varphi = \mathcal{G}^{-1} - \mathcal{G}_0^{-1} + \Sigma \quad ; \quad \boxed{\frac{1}{T}\frac{\delta \Phi[\mathcal{G}]}{\delta \mathcal{G}(1,2)} = \Sigma(2,1)}$$

Lecture #5 (90 min)

87. Source fields for many-body Green's functions

87.1 A simple example in classical stat. mech.

$$Z[h] = \mathrm{Tr}\left[e^{-\beta H - \int h(x)\, M(x)}\right]$$

$$\frac{\delta \ln Z[h]}{\delta h(x_1)} = \frac{1}{Z[h]}\, \mathrm{Tr}\left[e^{-\beta H - \int h(x)\, M(x)}\, M(x_1)\right]$$

$$= \langle M(x_1) \rangle \qquad \frac{\delta h(x)}{\delta h(x')} = \delta(x-x')$$

$$\frac{\delta^2 \ln Z[h]}{\delta h(x_1)\,\delta h(x_2)} = \langle M(x_1)\, M(x_2)\rangle - \langle M(x_1)\rangle\langle M(x_2)\rangle$$

c-number source fields for fermion bilinears

$$Z[\varphi] = \int \mathcal{D}\psi^+ \int \mathcal{D}\psi\; e^{\left[-S - \psi^+(\bar 1)\, \varphi(\bar 1, \bar 2)\, \psi(\bar 2)\right]}$$

$$(1) = (r_1, \tau_1, \sigma_1)$$

1 Bar $\to \int d^3x_1 \int_0^\beta d\tau_1 \sum_{\sigma_1}$ and $\dfrac{\delta \varphi(\bar 1, \bar 2)}{\delta \varphi(1,2)} = \delta(1-\bar 1)\,\delta(2-\bar 2)$

$$-\frac{\delta \ln Z[\varphi]}{\delta \varphi(2,1)} = -\langle T_\tau\, \psi(1)\, \psi^+(2)\rangle_\varphi = \mathcal{G}(1,2)$$

Let $S[\varphi] = S - \psi^+(\bar 1)\,\varphi(\bar 1, \bar 2)\,\psi(\bar 2)$

$$\frac{\delta \mathcal{G}(1,2)}{\delta \varphi(3,4)} = \frac{1}{Z[\varphi]} \int \mathcal{D}\psi^+ \int \mathcal{D}\psi \, e^{-S[\varphi]} \psi(1)\psi^+(2)\psi^+(3)\psi(4)$$

$$- \frac{1}{Z[\varphi]^2} \int \mathcal{D}\psi^+ \int \mathcal{D}\psi \, e^{-S[\varphi]} \psi(1)\psi^+(2)$$

$$\int \mathcal{D}\psi^+ \int \mathcal{D}\psi \, e^{-S[\varphi]} \psi^+(3)\psi(4)$$

$$= - \langle \psi^+(2)\psi(1)\psi^+(3)\psi(4) \rangle_\varphi + \mathcal{G}(1,2)_\varphi \mathcal{G}(4,3)_\varphi$$

(A) $$\frac{\delta^2 \ln Z[\varphi]}{\delta\varphi(1,2)\,\delta\varphi(3,4)} = -\mathcal{G}(2,1)_\varphi \mathcal{G}(4,3)_\varphi + \langle \psi^+(1)\psi(2)\psi^+(3)\psi(4)\rangle_\varphi$$

$$= - \frac{\delta \mathcal{G}(2,1)}{\delta\varphi(3,4)}$$

Dyson-Schwinger equation of motion

$$Z[\varphi, J, J^+] = \int \mathcal{D}\psi^+ \mathcal{D}\psi \, e^{-S[\varphi] - \psi^+(\bar{1})J(\bar{1}) - J^+(\bar{1})\psi(\bar{1})}$$

$$\int \mathcal{D}\psi^+ \mathcal{D}\psi \frac{\partial}{\partial \psi^+(1)} e^{-S[\varphi, J, J^+]} = 0$$

$$S[\varphi, J, J^+] = \psi^+(\bar{1})\left[-\mathcal{G}_0^{-1}(\bar{1},\bar{2}) + \varphi(\bar{1},\bar{2})\right]\psi(\bar{2})$$

$$+ \frac{1}{2} V(\bar{1},\bar{2})\psi^+(\bar{1})\psi^+(\bar{2})\psi(\bar{2})\psi(\bar{1})$$

$$-\frac{\partial}{\partial J(2)} \int \mathcal{D}\psi^+ \int d\psi \left[\frac{\partial S[\varphi]}{\partial \psi^+(1)} + J(1)\right] e^{-S[\varphi, J, J^+]} = 0$$

$$-\frac{1}{Z} \int \mathcal{D}\psi^+ \int \mathcal{D}\psi \left[- \frac{\partial S[\varphi]}{\partial \psi^+(1)} \psi^+(2) \right] e^{-S[\varphi]} = \delta(1-2)$$

anticomm de $J(2)$

$$+\left[-\mathcal{G}_0^{-1}(1,\bar{2})+\varphi(1,\bar{2})\right]\langle\psi(\bar{2})\,\psi^{+}(2)\rangle_{\varphi}$$

$$+\,V(1-\bar{3})\,\langle\psi^{+}(\bar{3})\,\psi(\bar{3})\,\psi(1)\,\psi^{+}(2)\rangle_{\varphi}=\delta(1-2)$$

$$\left(\mathcal{G}_0^{-1}(1,\bar{2})-\varphi(1,\bar{2})\right)\mathcal{G}(\bar{2},2)_{\varphi}=$$

$$\delta(1-2)-\underbrace{V(1-\bar{3})\,\langle\psi^{+}(\bar{3})\,\psi(\bar{3})\,\psi(1)\,\psi^{+}(2)\rangle_{\varphi}}$$

$$=\Sigma(1,\bar{2})\,\mathcal{G}(\bar{2},2)$$

36.3 Four-point function from functional derivatives

$$\frac{\delta}{\delta\varphi}\left(\mathcal{G}^{-1}\mathcal{G}\right)=0\quad\rightarrow\quad\frac{\delta\mathcal{G}^{-1}}{\delta\varphi}\mathcal{G}+\mathcal{G}^{-1}\frac{\delta\mathcal{G}}{\delta\varphi}=0$$

$$1 \longrightarrow 2$$

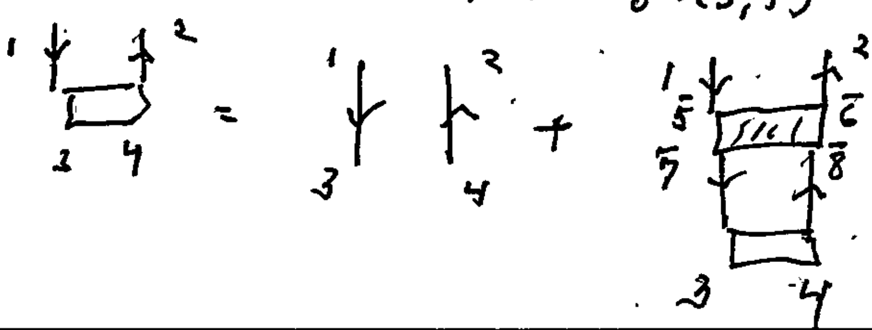
$$\mathcal{G}(1,2)$$

$$\frac{\delta\mathcal{G}}{\delta\varphi}=-\mathcal{G}\,\frac{\delta\mathcal{G}^{-1}}{\delta\varphi}\,\mathcal{G}$$

$$=-\mathcal{G}\,\frac{\delta\varphi}{\delta\varphi}\,\mathcal{G}+\mathcal{G}\,\frac{\delta\Sigma}{\delta\varphi}\,\mathcal{G}$$

$$\frac{\delta\mathcal{G}(1,2)}{\delta\varphi(3,4)}=\mathcal{G}(1,\bar{2})\,\frac{\delta\varphi(\bar{2},\bar{3})}{\delta\varphi(3,4)}\,\mathcal{G}(\bar{3},2)$$

$$+\,\mathcal{G}(1,\bar{5})\,\frac{\delta\Sigma(\bar{5},\bar{6})}{\delta\mathcal{G}(\bar{7},\bar{8})}\,\frac{\delta\mathcal{G}(\bar{7},\bar{8})}{\delta\varphi(3,4)}\,\mathcal{G}(\bar{6},2)$$



Irreducible particle-hole vertex

36.4 Self-energy from functional derivatives

$$\Sigma(1,3) = -V(1-\bar{2})\left[\frac{\delta \mathcal{G}(1,\bar{4})_\varphi}{\delta\varphi(\bar{2}^+,\bar{2})} - \mathcal{G}(\bar{2},\bar{2}^+)_\varphi\, \mathcal{G}(1,\bar{4})_\varphi\right]\mathcal{G}^{-1}(\bar{4},3)_\varphi$$

$$= -V(1-\bar{2})\,\langle T_\tau[\psi^+(\bar{2}^+)\psi(\bar{2})\psi(1)\psi^+(\bar{4})\rangle\, \mathcal{G}^{-1}(\bar{4},3)$$

$$\Sigma(1,3) = -V(1-\bar{2})\Big[\mathcal{G}(1,\bar{2}^+)\mathcal{G}(\bar{2},\bar{4}) - \mathcal{G}(1,\bar{7})\frac{\delta\Sigma(\bar{7},\bar{8})}{\delta\varphi(\bar{2}^+,\bar{2})}$$

$$- \mathcal{G}(\bar{2},2^+)\mathcal{G}(1,\bar{4}) \qquad \mathcal{G}(\bar{8},\bar{4})\Big]\mathcal{G}^{-1}(\bar{4},3)$$

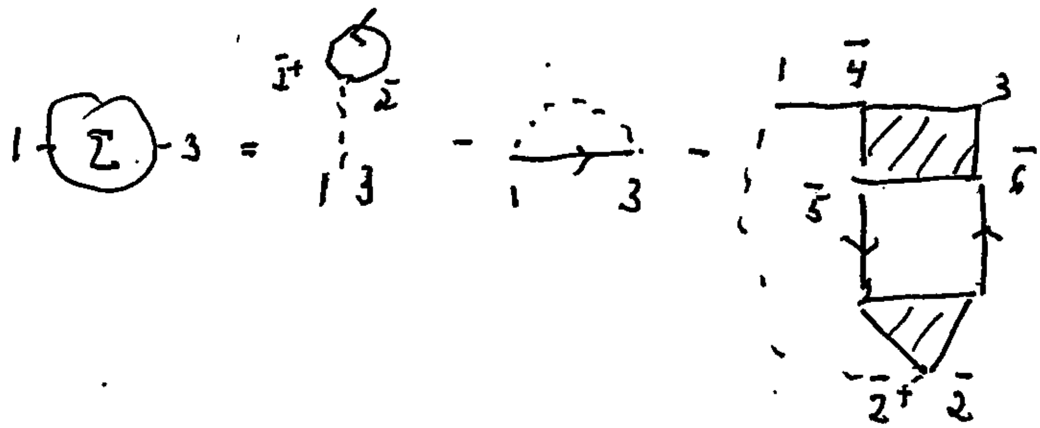
$$= -V(1-\bar{2})\left[\frac{\delta\mathcal{G}}{\delta\varphi} - \frac{\mathcal{G}}{\mathcal{G}}\right]\mathcal{G}^{-1}$$

$$= -V(1-\bar{2})\left[\mathcal{G}_{\wedge}\mathcal{G} - \frac{\mathcal{G}}{\mathcal{G}} + \mathcal{G}\,\frac{\delta\Sigma}{\delta\mathcal{G}}\,\mathcal{G}\right]\mathcal{G}^{-1}$$

$$\frac{\delta\mathcal{G}}{\delta\varphi}$$

$$= -V(1-\bar{2})\,\mathcal{G}(1,3) + V(1-\bar{2})\,\frac{\delta(1-3)}{\mathcal{G}(\bar{2}^+,\bar{2})}$$

$$- V(1-\bar{2})\left[\mathcal{G}(1,\bar{4})\,\frac{\delta\Sigma(\bar{4},3)}{\delta\mathcal{G}(\bar{5},\bar{6})}\,\frac{\delta\mathcal{G}(\bar{5},\bar{6})}{\delta\varphi(\bar{2}^+,\bar{2})}\right]$$



3.6 Irreducibility

Green function = one-particle reducible

Self-energy = one-particle irreducible

$$\frac{1}{\mathcal{G}_0^{-1} - \Sigma} = \frac{1}{1 - \Sigma \mathcal{G}_0} \mathcal{G}_0 = \mathcal{G}_0 + \mathcal{G}_0 \Sigma \mathcal{G}_0 + \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \mathcal{G}_0 + \cdots$$

Chap. 4 Luttinger-Ward functional

$$\frac{\delta^2 \ln Z[\varphi]}{\delta\varphi(1,2)\,\delta\varphi(3,4)} = \frac{\delta^2 \ln Z[\varphi]}{\delta\varphi(3,4)\,\delta\varphi(1,2)}$$

$$\frac{\delta \mathcal{G}(2,1)}{\delta\varphi(3,4)} = \frac{\delta \mathcal{G}(4,3)}{\delta\varphi(1,2)}$$

$$\left[\mathcal{G}^{-1}\frac{\delta \mathcal{G}}{\delta\varphi}\mathcal{G}^{-1} - \frac{\delta\Sigma}{\delta\mathcal{G}}\frac{\delta\mathcal{G}}{\delta\varphi}\right]$$

↑ must be symmetric like this

$\Rightarrow \exists$ functional

72. Luttinger-Ward and related functionals

Free energy $F[\varphi] = -T\ln Z[\varphi]$

$$\boxed{\frac{1}{T}\frac{\delta F[\varphi]}{\delta\varphi(1,2)} = \mathcal{G}(2,1)}$$

Free-energy at $\varphi = 0$
Prefer to work with observable $\mathcal{G}$.

$$\boxed{\Omega[\mathcal{G}] = F[\varphi] - \mathrm{Tr}[\varphi\mathcal{G}]} = \text{Kadanoff-Baym functional.}$$

(assumes local convexity)

$$\mathrm{Tr}[\varphi\mathcal{G}] = T\varphi(\bar{1},\bar{2})\mathcal{G}(\bar{2},\bar{1})$$

$$= T\sum_{ik_n}\sum_{k}\varphi(k,ik_n)\mathcal{G}(k,ik_n)$$

Like all Legendre transforms

$$\boxed{\frac{1}{T}\frac{\delta\Omega}{\delta\mathcal{G}(1,2)} = -\varphi(2,1)} = \mathcal{G}^{-1}(2,1) - \mathcal{G}_0^{-1}(2,1) + \Sigma(2,1)$$

↑ Eq. of motion
$\Rightarrow$ at equilibrium $\varphi = 0$ and Dyson satisfied

Proof:

$$\frac{1}{T}\frac{\delta\Omega}{\delta\mathcal{G}} = \frac{1}{T}\frac{\delta F}{\delta\varphi}\frac{\delta\varphi}{\delta\mathcal{G}} - \frac{\delta}{\delta\mathcal{G}}[\varphi\mathcal{G}]$$

$$= \mathcal{G}\frac{\delta\varphi}{\delta\mathcal{G}} - \mathcal{G}\frac{\delta\varphi}{\delta\mathcal{G}} - \varphi$$

76. Constraining field

$$dE = TdS - pdV \Rightarrow p = -\left(\frac{\partial E}{\partial V}\right)_S$$

$$dF = -SdT - pdV \Rightarrow p = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$\left.\frac{\partial \Omega_\lambda}{\partial \lambda}[\mathcal{G}]\right|_{\mathcal{G}} = \left.\frac{\partial F_\lambda[\varphi]}{\partial \lambda}\right|_{\varphi} = \frac{1}{\lambda}\langle \lambda V\rangle_\lambda$$

$$\boxed{\Omega_\lambda[\mathcal{G}]} = \Omega_{\lambda=0}[\mathcal{G}] + \int_0^1 d\lambda \, \frac{1}{\lambda}\langle \lambda V\rangle_\lambda$$

$$= \underbrace{F[\varphi_0] - \mathrm{Tr}[\varphi_0 \mathcal{G}]}_{\varphi_0 \text{ to give correct } \mathcal{G}} + \underbrace{\Phi[\mathcal{G}]}_{\text{Luttinger-Ward}}$$

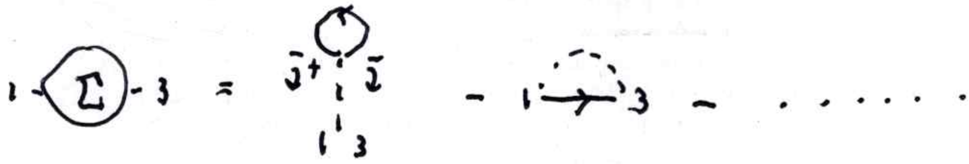
$$= \boxed{\mathrm{Tr}\left[\ln(-\mathcal{G})\right] - \mathrm{Tr}\left[(\mathcal{G}_0^{-1} - \mathcal{G}^{-1})\mathcal{G}\right] + \Phi[\mathcal{G}]}$$

$$\frac{1}{T}\frac{\delta \Omega}{\delta \mathcal{G}} = -\varphi = \mathcal{G}^{-1} - \mathcal{G}_0^{-1} + \Sigma \quad \text{from equ. of motion}$$

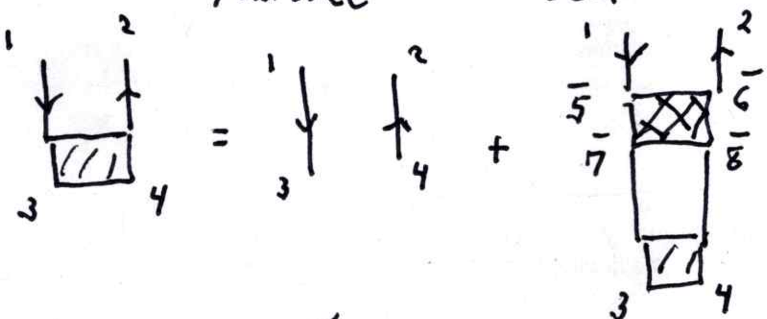
$$\Rightarrow \frac{1}{T}\frac{\delta \Phi}{\delta \mathcal{G}} = \Sigma \qquad \boxed{\frac{1}{T}\frac{\delta \Phi}{\delta \mathcal{G}(1,2)} = \Sigma(2,1)}$$

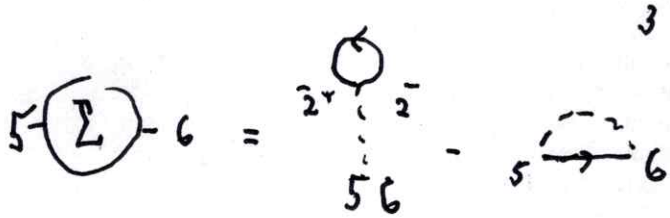
Lecture #6 (90 min)

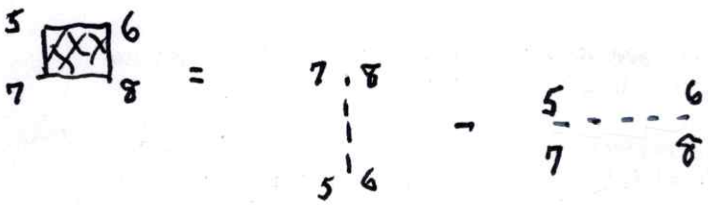
Hartree-Fock and RPA in space-imaginary time

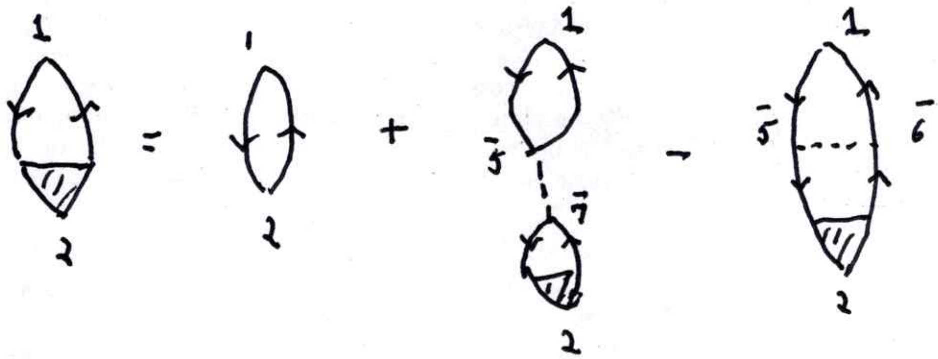


Hartree Fock









$$\frac{\delta \mathcal{G}(1,1^+)}{\delta \varphi(2^+,2)} = -\langle T_\tau \psi^+(1^+)\psi(1)\psi^+(2^+)\psi(2)\rangle + \mathcal{G}(1,1^+)\mathcal{G}(2,2^+)$$

$$\langle T_\tau\, n(1)\, n(2)\rangle = -\sum_{\sigma_1\sigma_2} \frac{\delta \mathcal{G}(1,1^+)}{\delta \varphi(2^+,2)} + n^2$$

$$\langle T_\tau\, (n(1)-n)(n(2)-n)\rangle = \chi_{nn}(1-2) = -\sum_{\sigma_1\sigma_2} \frac{\delta \mathcal{G}(1,1^+)}{\delta \varphi(2^+,2)}$$

Hartree-Fock + RPA in momentum - Matsubara frequency.

$$\chi^0_{nn}(q) = -\int d(x_1-x_2)\int_0^\beta d(\tau_1-\tau_2)\, e^{-i\vec{q}\cdot(\vec{x}_1-\vec{x}_2)+iq_n(\tau_1-\tau_2)} \sum_\sigma \mathcal{G}_\sigma(1-2)\,\mathcal{G}_\sigma(2-1)$$

$\uparrow$ $\vec{q}, iq_n$

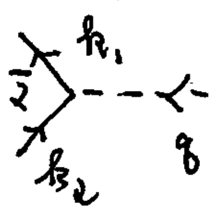
Convolution

$$= -\int \frac{d^3k}{(2\pi)^3}\, T\sum_{n=-\infty}^{\infty} \mathcal{G}(k)\,\mathcal{G}(k+q)$$

N.B. $\langle T_\tau\, n(1)\, n(2)\rangle = \langle T_\tau\, n(2)\, n(1)\rangle$

+ periodic $\Rightarrow$ $q_n = (2n)(\pi T)$

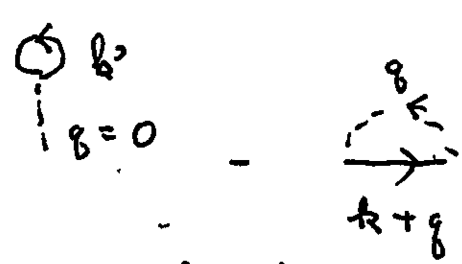
Generally at vertex

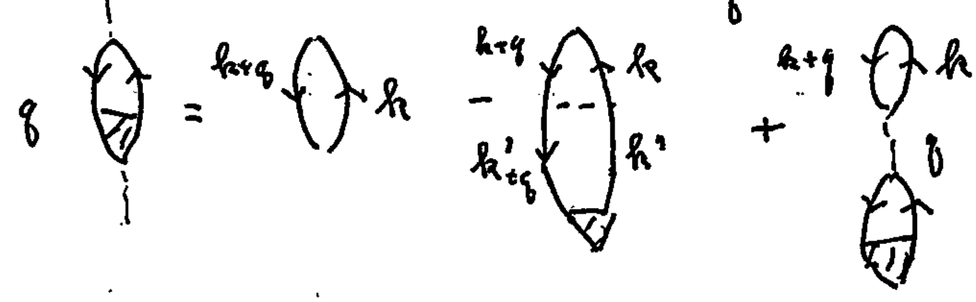


$$\int dx_2 \int_0^\beta d\tau_2\, e^{-i(k_1-k_2+q)x_2}\, e^{i(k_{1,n}-k_{2,n}+q_n)\tau_2}$$

$$= (2\pi)^3\, \delta(k_1-k_2+q)\, \beta\, \delta_{k_{1n}-k_{2n},\, q_n}$$

$\Sigma(k) =$





Non-interacting limit - Lindhard function

$$\chi^0_{nn}(q, iq_n) = -\sum_\sigma \int \frac{d^3k}{(2\pi)^3} \, T \sum_{ik_n} \mathcal{G}^0_\sigma(k+q, ik_n + iq_n) \, \mathcal{G}^0_\sigma(k, ik_n)$$

$$= -2 \int \frac{d^3k}{(2\pi)^3} \, \underbrace{T \sum_{ik_n} \frac{1}{ik_n + iq_n - \xi_{k+q}} \, \frac{1}{ik_n - \xi_k}}$$

$$T \sum_{ik_n} \left(\frac{1}{ik_n - \xi_k} - \frac{1}{ik_n + iq_n - \xi_{k+q}} \right) \frac{1}{iq_n - \xi_{k+q} + \xi_k}$$

$$\boxed{\chi^0_{nn}(q, iq_n) = -2 \int \frac{d^3k}{(2\pi)^3} \, \frac{f(\xi_k) - f(\xi_{k+q})}{iq_n - \xi_{k+q} + \xi_k}}$$ Lindhard function

↑

analytic continuation

N.B. • Particle - hole

• Lehmann representation.

$\chi_{nn}(q)$ in RPA approximation

$$\chi_{nn}(q) = \chi^0_{nn}(q) - \chi^0_{nn}(q)\, V_q\, \chi_{nn}(q) \qquad \text{Bethe-Salpeter}$$

$$\boxed{\chi_{nn}(q) = \frac{\chi^0_{nn}(q)}{1 + V_q \chi^0_{nn}(q)} = \frac{1}{\chi^{0\,-1}_{nn}(q) + V_q}}$$

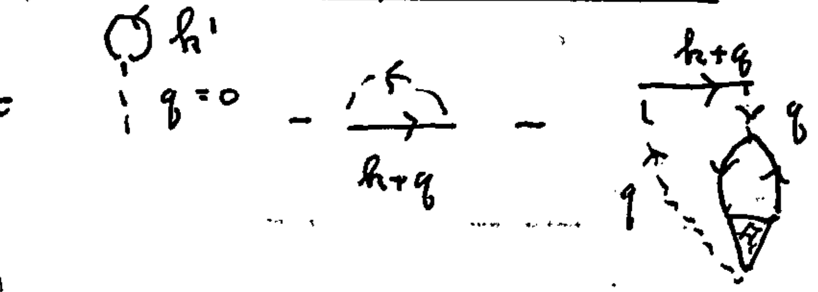
cf $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \Sigma \mathcal{G}$; $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} - \Sigma$

- V_q like self-energy in p-h. channel
  ↳ more generally irreducible vertex

- $\chi_{nn} = \chi^0_{nn} - \chi^0_{nn} V_q \chi^0_{nn} + \chi^0_{nn} \underbrace{V_q \chi^0_{nn} V_q}_{\sim 1/q^4} \chi^0_{nn} + \dots$

$\sim 1/q^4 \Rightarrow$ must sum. displace poles.

Second step GW, curing Hartree-Fock

$\Sigma_k =$ 

$$= -\int \frac{d^3q}{(2\pi)^3}\, T \sum_{iq_n} \underbrace{V_q \left[1 - \frac{V_q \chi^0_{nn}}{1 + V_q \chi^0_{nn}}\right]}\, \mathcal{G}^0(k+q, ik_n + iq_n)$$

↳ $\dfrac{V_q}{1 + V_q \chi^0_{nn}}$ screening = dielectric constant

Hubbard model in the footsteps of the electron gas

Pauli and Mermin-Wagner : No

TPSC to cure

Response functions for spin and charge

$$\hat{H} = -\sum_{ij}\sum_{\sigma} t_{ij}\left(c^{+}_{i\sigma} c_{j\sigma}\right) + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

$$-\frac{\delta \mathcal{G}_{\sigma}(1,1^{+})}{\delta \varphi_{\sigma'}(2^{+},2)} = \langle T_{\tau}\, n_{\sigma}(1)\, n_{\sigma'}(2)\rangle - \langle n_{\sigma}(1)\rangle\langle n_{\sigma'}(2)\rangle$$

$\uparrow$ spin label explicit

$$\chi_{ch} = -\sum_{\sigma\sigma'} \frac{\delta \mathcal{G}_{\sigma}}{\delta \varphi_{\sigma'}} \equiv \chi_{\uparrow\uparrow} + \chi_{\uparrow\downarrow} + \chi_{\downarrow\uparrow} + \chi_{\downarrow\downarrow}$$

$$\chi_{sp} = \chi_{\uparrow\uparrow} - \chi_{\uparrow\downarrow} - \chi_{\downarrow\uparrow} + \chi_{\downarrow\downarrow}$$

$$s^{T} = (1,1) \qquad a^{T} = (1,-1)$$

$$\chi_{ch} = s^{T}\begin{pmatrix} \chi_{\uparrow\uparrow} & \chi_{\uparrow\downarrow} \\ \chi_{\downarrow\uparrow} & \chi_{\downarrow\downarrow}\end{pmatrix} s \qquad \chi_{sp} = a^{T}\overleftrightarrow{\chi}\, a$$

$$0 = s^{T}\chi\, a = a^{T}\chi\, s \qquad s \otimes s^{T} = \begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}$$

$$a \otimes a^{T} = \begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}$$

$$s^T \chi s = -2 \mathcal{G}\mathcal{G} + \mathcal{G}\, s^T \frac{\overleftrightarrow{\delta \Sigma}}{\delta \mathcal{G}} \left(\frac{a \otimes a^T + s \otimes s^T}{2} \right) \chi_s \mathcal{G}$$

$$\boxed{\begin{aligned} \chi_{ch} &= -2\mathcal{G}\mathcal{G} + \mathcal{G}\left[\left(\frac{\delta \Sigma_\uparrow}{\delta \mathcal{G}_\uparrow} + \frac{\delta \Sigma_\downarrow}{\delta \mathcal{G}_\downarrow}\right)\right] \chi_{ch} \mathcal{G} \\ \chi_{sp} &= -2\mathcal{G}\mathcal{G} - \mathcal{G}\left[\left(\frac{\delta \Sigma_\uparrow}{\delta \mathcal{G}_\downarrow} - \frac{\delta \Sigma_\uparrow}{\delta \mathcal{G}_\uparrow}\right)\right] \chi_{sp} \mathcal{G} \end{aligned}}$$

Hartree-Fock and RPA

$$\Sigma_\sigma (1,\bar{1})_\varphi \mathcal{G}_\sigma (\bar{1},2)_\varphi = -U \langle T_\tau \psi^+_{-\sigma}(1^+) \psi_{-\sigma}(1) \psi_\sigma(1) \psi^+_\sigma(2) \rangle_\varphi$$

$$= -U \left[\frac{\delta \mathcal{G}_\sigma (1,2)}{\delta \varphi_{-\sigma}(1^+ 1)} - \mathcal{G}_{-\sigma}(1,1^+)_\varphi \mathcal{G}_\sigma (1,2)_\varphi \right]$$

$$\Sigma^H_\sigma (1,\bar{1})_\varphi \mathcal{G}^H_\sigma (\bar{1},2)_\varphi = U \mathcal{G}^H_{-\sigma}(1,1^+)_\varphi \mathcal{G}^H (1,2)_\varphi$$

$$\Sigma^H_\sigma (1,2) = U \mathcal{G}^H_{-\sigma}(1,1^+)\, \delta(1-2)$$

$$\frac{\delta \Sigma^H_\uparrow (1,2)}{\delta \mathcal{G}^H_\uparrow (3,4)} = 0 \qquad \frac{\delta \Sigma^H_\uparrow (1,2)}{\delta \mathcal{G}^H_\downarrow (3,4)} = U\, \delta(1-2)\, \delta(1-3)\, \delta(2-4)$$

$$\boxed{\chi_{ch} = \frac{\chi_0}{1 + \frac{1}{2} U \chi_0} \quad ; \quad \chi_{sp} = \frac{\chi_0}{1 - \frac{1}{2} U \chi_0}}$$

RPA and violation of the Pauli principle

$$\frac{T}{N}\sum_q \sum_{iq_n} \chi_{sp}(q, iq_n) = \langle (n_\uparrow - n_\downarrow)^2 \rangle = n - 2\langle n_\uparrow n_\downarrow \rangle$$

$$\frac{T}{N}\sum_q \sum_{iq_n} \chi_{ch}(q, iq_n) = \langle (n_\uparrow + n_\downarrow)^2 \rangle - \langle (n_\uparrow + n_\downarrow) \rangle^2$$

$$= n + 2\langle n_\uparrow n_\downarrow \rangle - n^2$$

$$\frac{T}{N}\sum_q \left(\frac{\chi_0}{1 - \frac{U}{2}\chi_0} + \frac{\chi_0}{1 + \frac{U}{2}\chi_0} \right) = 2n - n^2$$

satisfied only to first order in U

Mermin-Wagner

$$q^2 \langle \phi_q \phi_{-q} \rangle = \frac{k_B T}{2} \qquad \langle \phi_i^2 \rangle = \int_{-\infty}^{\infty} \frac{d^2 q}{(2\pi)^d} \frac{k_B T}{q^2} = \infty$$

Two particle self-consistent

$$\Sigma_\sigma^{(1)}(1,\bar{1})_\varphi \, \mathcal{G}_\sigma^{(1)}(\bar{1},2)_\varphi = A_\varphi \, \mathcal{G}_{-\sigma}^{(1)}(1,1^+)_\varphi \, \mathcal{G}_\sigma^{(1)}(1,2)_\varphi$$

$$\Sigma_\sigma^{(1)}(1,\bar{1})_\varphi \, \mathcal{G}_\sigma^{(1)}(\bar{1},1^+)_\varphi = U\langle n_\uparrow n_\downarrow \rangle_\varphi$$

$$\Sigma_\sigma^{(1)}(1,2)_\varphi = A_\varphi \, \mathcal{G}_{-\sigma}^{(1)}(1,1^+)\,\delta(1-2)$$

$$\left.\frac{\delta \Sigma_\uparrow^{(1)}(1,2)_\varphi}{\delta \mathcal{G}_\downarrow^{(1)}(3,4)_\varphi}\right|_{\varphi=0} - \frac{\delta \Sigma_\downarrow(1,2)}{\delta \mathcal{G}_\uparrow(3,4)} = U_{sp}\,\delta(1-2)\,\delta(3-1)\,\delta(2-4)$$

$$U_{sp} = \frac{U\langle n_\uparrow n_\downarrow \rangle}{\langle n_\uparrow \rangle \langle n_\downarrow \rangle}$$

$$\frac{T}{N}\sum_q \frac{\chi^{(1)}}{1-\frac{1}{2}U_{sp}\chi^{(1)}} = n - 2\langle n_\downarrow n_\uparrow\rangle$$

gives U_{sp}

$$\frac{T}{N}\sum_q \frac{\chi^{(1)}}{1+\frac{1}{2}U_{ch}\chi^{(1)}} = n + 2\langle n_\uparrow n_\downarrow\rangle - n^2$$

TPSC Second step. Improving the self-energy

Analogy with GW

Longitudinal + transverse fluctuations.

$$\boxed{\Sigma_\sigma^{(2)}(k) = U n_{-\sigma} + \frac{U}{8}\frac{T}{N}\sum_q \left[3U_{sp}\chi_{sp}(q) + U_{ch}\chi_{ch}(q)\right] \mathcal{G}^{(1)}(k+q)}$$

Internal accuracy check

$$\mathrm{Tr}\left[\Sigma^{(2)}\mathcal{G}^{(1)}\right] = U\langle n_\uparrow n_\downarrow\rangle$$

Generalizations

TPSC+ TPSC+ DMFT